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NON-LINEAR GAUGE-FIXING IN THE STANDARD MODEL AT ONE-LOOP LEVEL

Master's thesis
(Theoretical physics and astrophysics studies program)

Vilnius 2010

Contents

1 Introduction

The currently accepted theory of elementary particles and fundamental interactions — the Standard Model (SM), introduced and shaped in the 70s, has been very successful at explaining all observed and measured effects up till now. All currently observed elementary particles — leptons and hadrons — and the fundamental interactions — electromagnetic, weak and strong — have been successfully incorporated into the theory and thoroughly tested in particle colliders. To date, all experimental data has agreed with the predictions of the Standard Model with high precision.

The theory has also been successful at predicting new phenomena that would later be verified experimentally. A notable example is the Z gauge boson and the neutral current interactions due to it, which emerge naturally from the Glashow-Salam-Weinberg model of electroweak unification. The only prediction yet to be verified experimentally is the existence of the famous Higgs boson, which despite intensive searches at the CERN and Fermilab particle colliders has yet to be found. The situation may change with the completion and launch of the Large Hadron Collider (LHC) at CERN in the near future, which may reach energies as high as 14 TeV[1].

The Standard Models falls short of being a complete theory of fundamental interactions of nature (Theory of Everything) because of the exclusion of gravity. Furthermore it is inconsistent with recent observations of neutrino flavor oscillations, which imply non zero neutrino masses. It is also well established that the Standard Model is more of a perturbative theory — as the energies become higher the theory breaks down and new physics should emerge. This could already be happening at the scale of 1000 GeV. Various ideas have been proposed to account for this new physics, e.g. extending the Standard Model to include supersymmetry — the Minimal Supersymmetric Standard Model (MSSM). Radically new theories have also been proposed, the most famous one being String Theory. However, there are currently no experimental data to test either of these new theories, thus the Standard Model remains the theory of choice for particle physics.

Even though the Standard Model is a very complicated theory, the process of calculating cross sections and decay rates becomes very simple in principle with the introduction of *Feynman diagrams*, which are basically pictorial representations of the process being considered. After drawing all diagrams for a given process, algebraic expressions are constructed from the diagrams using a set of rules, called the *Feynman rules*. The expressions are then calculated straightforwardly, summed up, squared and expressed as cross sections or decay rates. Though the procedure is relatively simple, usually, the amount of algebra to be done is huge, especially if we want to consider quantum corrections by including loop diagrams. Therefore, the calculations are usually done using computer software packages, one notable example being FeynArts.

Another notable problem in calculating scattering amplitudes is that some expressions in higher order diagrams may contain divergences. They introduce not only conceptual problems in applying quantum field theory, but also computational difficulties. And even though the conceptual issues have eventually been resolved and understood, the computational difficulties are still a major headache when doing calculations by hand. The process of getting rid of and interpreting the divergences is called *renormalization*. The renormalization procedure yet again encourages the use of various software packages, which nowadays are able to perform this, most of the time technical task, almost automatically.

With complicated calculations comes the requirement to be able to check the validity of the results. Gauge theories provide a useful tool in doing that by gauge-fixing. Gauge-fixing is a procedure in gauge theories analogous to gauge-fixing in classical electrodynamics, where extra degrees of freedom have to be constrained and the result should not depend on how we constrain the theory. The consistency check is even more evident when Feynman diagrams enter the picture, since expressions for a particular diagram in a gauge-fixed theory will usually contain gauge parameters, hence be gauge dependant. The miracle happens when we add up the diagrams to get a physical result — all the gauge parameters cancel. The usual choice is to introduce one gauge parameter per gauge field, the so called *linear gauge-fixing*. In this paper we will look at implementing *non-linear gauge-fixing*, i.e. putting constraints on the fields, which are non-linear, resulting in even more gauge parameters, thus ensuring a better consistency check for the calculation software.

The aim of this paper is to implement non-linear gauge-fixing in the FeynArts package and check the validity of the implementation at tree and one-loop levels in the Standard Model. We start by discussing the tools needed in order to construct the Standard Model, then proceed by constructing it and discussing the concepts of gauge-fixing and renormalization. We then discuss how non-linear gauge-fixing is implemented in the FeynArts package and show some explicit calculations within the non-linearly gauge-fixed Standard Model. The goal of the calculations is to check whether the implementation is correct and whether FeynArts performs the calculations correctly, i.e. whether we get any dependence of some cross section on the gauge-fixing parameters. One of the most important aims of this paper is to check whether the implementation behaves correctly at one-loop level, where the renormalization concept enters. We try to calculate various processes and determine the limits where the implementation breaks down.

2 Theoretical background

Before constructing the Standard Model and discussing non-linear gauge-fixing, we have to introduce the relevant tools. Mathematically the Standard Model is a gauge quantum field theory with the local gauge group[2]

$$G_{loc} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y. \tag{2.1}$$

This symmetry group is a non-abelian group — gauge theories of this kind are called Yang-Mills theories. The Lagrangian of a Yang-Mills theory has a definite form and is called a Yang-Mills Lagrangian. In the following sections we will give brief introductions to the above mentioned concepts. Since renormalization plays an important role in this paper, we discuss it a separate section.

2.1 Gauge theories

2.1.1 Symmetry and gauge theories

Almost all modern physical theories one way or the other study symmetries of the system in question. No exception is quantum field theory. E.g. take the Lagrangian of a particle satisfying the Dirac equation with electromagnetic interactions[3]

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.2}$$

We can perform a global transformation of the field ψ by arbitrarily changing it's phase — this leaves the Lagrangian invariant. The theory is said to have a *global* internal symmetry of $U(1)$. Remarkably, it turns out that we can perform a *local* transformation on the fields

$$\psi(x) \to e^{i\Lambda(x)}\psi(x) \tag{2.3}$$

and

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\Lambda(x) \tag{2.4}$$

which implies

$$F_{\mu\nu} \to F_{\mu\nu}. \tag{2.5}$$

The latter is of course the well known gauge transformation of the classical electromagnetic field. Thus the Dirac Lagrangian is said to have an internal local symmetry of $U(1)$.

Discovered almost by accident, this local symmetry turns out to have fundamental significance. In fact, from the modern point of view, (2.2) is considered a consequence of (2.3) and (2.4) and not the other way around. All modern quantum field theories are built by promoting some subgroup of the global symmetry group to a local symmetry group. Such theories are called *gauge theories*. As it turns out, all the quantum field theories describing the fundamental interactions of nature are gauge theories.

2.1.2 Yang-Mills Lagrangian

We can now generalize the Lagrangian in (2.2) and build a general Yang-Mills Lagrangian, which is the backbone of any gauge field theory, from general group theory principles. We start with a multiplet of fields, transforming as some representation R of an $SU(N)$ gauge group G:

$$\psi(x) \equiv \begin{pmatrix} \psi_2(x) \\ \psi_1(x) \\ \vdots \\ \psi_N(x) \end{pmatrix}. \tag{2.6}$$

The transformation law is then given by

$$\psi(x) \to V(x)\psi(x), \tag{2.7}$$

where $V(x)$ is a local $N \times N$ transformation matrix. Group theory suggests that any arbitrary group element can be expressed as a linear combination of the group's *generators*, i.e. the infinitesimal field transformations can be expressed as

$$\delta\psi_i(x) = i\alpha^a(x)t^a_{ij}\psi_j(x), \tag{2.8}$$

where $\alpha^a(x)$ are the infinitesimal transformation parameters and t^a is the a'th generator of the representation R. The generators also satisfy the commutation relation

$$[t^a, t^b] = if^{abc}t^c, \tag{2.9}$$

f^{abc} being the structure constants of the group G.

Since the derivatives of the fields are usually gauge dependant, a *covariant derivative* is defined as

$$D_\mu = \partial_\mu - igA^a_\mu t^a \tag{2.10}$$

by introducing a gauge field A^a_μ for each generator and a coupling constant g, which can be chosen independently for each simple or $U(1)$ subgroup of the gauge group. Since our gauge group G is $SU(N)$, we have only one coupling constant. In the most general case, we would have to sum over all simple and $U(1)$ subgroups of G with different coupling constants g^r. The covariant derivative is gauge independent when the gauge fields have transformation laws defined as

$$\delta A^a_\mu = \frac{1}{g}\partial_\mu\alpha^a(x) + f^{abc}A^b_\mu\alpha^b(x). \tag{2.11}$$

A field strength tensor can then be defined as

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu, \tag{2.12}$$

which transforms under a gauge transformation as

$$\delta F^a_{\mu\nu} = -f^{abc}\alpha^b(x)F^c_{\mu\nu}. \tag{2.13}$$

The last terms in (2.11), (2.12) and (2.13) are non-abelian terms and vanish in abelian theories.

We can now write down the most general non-abelian gauge independant Lagrangian for Dirac fields

$$\mathcal{L}_{YM} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{2.14}$$

The last term in the above Lagrangian when considered alone is called a *Yang-Mills Lagrangian* — any theory described by this Lagrangian is a Yang-Mills theory. We can analyze the (2.14) Lagrangian better with the covariant derivative written out explicitly and by introducing field currents

$$j^{\mu a} = \bar{\psi}\gamma^\mu t^a \psi, \tag{2.15}$$

thus we have

$$\mathcal{L}_{YM} = \bar{\psi}i\gamma^\mu \partial_\mu \psi + gj^{\mu a}A^a_\mu - m\bar{\psi}\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{2.16}$$

The first term is just the kinetic energy of the fields $\psi(x)$, the second term describes the coupling of these fields to the gauge fields A^a_μ. The third term is the mass term with m being the field mass matrix. Finally the last term describes the energy of the gauge fields. We see that the fields $\psi(x)$ talk to each other only via the gauge fields, quanta of which can be identified as interaction carrying particles. Since there are no mass terms for the gauge fields, we conclude that they are massless.

Looking back at the Dirac Lagrangian (2.2) we can easily see that it is a Yang-Mills Lagrangian. Thus quantum electrodynamics (QED) is a gauge quantum field theory with the gauge group $U(1)$. The field $\psi(x)$ is the spin $1/2$ electron matter field with mass m. The only gauge field in this theory is the spin 1 massless field A_μ, better known as the electromagnetic field with it's quantum, the photon. The coupling constant is α, the fine structure constant, which is equal to $\approx 1/137$.

2.1.3 Fadeev-Popov procedure

In order to calculate physical quantities like cross sections and decay rates in gauge theories using Feynman diagrams, one must first construct the Feynman rules of the theory, which is described by some Lagrangian $\mathcal{L}$. Propagators and interaction vertices can be read off from the path integral

$$Z = \int \mathcal{D}A\,\mathcal{D}\psi\, e^{i\int \mathcal{L}(A,\psi)}. \tag{2.17}$$

However, this path integral runs over infinitely many configurations which differ only by gauge symmetry, thus are physically equivalent. Therefore, calculations using this path integral fail. Some way of eliminating this redundant symmetry is needed.

In order to factor out the gauge equivalent field configurations from (2.17) we adopt the *Fadeev-Popov procedure* [4], which is basically restricting the integral to physical configurations only by inserting a delta function $\delta(G(A))$ in (2.17). Formally this is done by inserting an identity expressed as

$$1 = \int \mathcal{D}\alpha(x)\delta(G(A^\alpha)) \det \left(\frac{\delta G(A^\alpha)}{\delta\alpha} \right) \tag{2.18}$$

into the integral, resulting in the expression

$$Z = \Delta \int \mathcal{D}\alpha(x) \int \mathcal{D}A\,\mathcal{D}\psi\, e^{i\int \mathcal{L}(A,\psi)}\delta(G(A^\alpha)), \tag{2.19}$$

where

$$\Delta \equiv \det \left(\frac{\delta G(A^\alpha)}{\delta\alpha} \right) \tag{2.20}$$

is the Jacobian determinant that arises during the change of variables and A^α is the gauge transformed gauge field. Here and further on we implicitly leave out normalization constants, which are irrelevant for Feynman rules. The functions $G(A)$ are constraining functions in a sense that we set them equal to zero as a gauge-fixing condition, e.g.

$$G(A) \equiv \partial_\mu A^\mu = 0 \tag{2.21}$$

corresponds to the Lorentz gauge.

Since A and A^α are physically equivalent, we can rename A^α to A in (2.19). In the end, the delta function restricts the integral over A to physical configurations and the integral over different gauges factors out as an infinite multiplicative factor. We specify the gauge-fixing function to be a *linear* functional of the gauge fields

$$G(A) = \partial^\mu A_\mu(x) - \omega(x), \tag{2.22}$$

where $\omega(x)$ is any scalar function.

Since we can choose $\omega(x)$ to be anything we like, we can stick the gauge function into (2.19) and integrate over $\omega(x)$ with a Gaussian weighting function centered on $\omega = 0$:

$$Z = \Delta \int \mathcal{D}\omega \exp\left[-i \int d^4x \frac{\omega^2}{2\xi} \right] \int \mathcal{D}\alpha \int \mathcal{D}A\,\mathcal{D}\psi\, e^{i\int \mathcal{L}(A,\psi)}\delta(\partial^\mu A_\mu - \omega)), \tag{2.23}$$

where ξ is any arbitrary constant. Integrating the expression gives

$$Z = \Delta \int \mathcal{D}\alpha \int \mathcal{D}A\,\mathcal{D}\psi\, e^{i\int \mathcal{L}(A,\psi)} \exp\left[-i \int d^4x \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \right], \tag{2.24}$$

effectively adding an extra piece to the Lagrangian

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2. \tag{2.25}$$

The parameter ξ is a *gauge-fixing constant*, meaning that no physical quantity should depend on it and we are free to choose it freely during calculations — it should cancel in the end. We see that gauge-fixing implicitly provides a mechanism for checking the correctness of our calculations.

2.1.4 Ghosts

The Fadeev-Popov procedure, no matter how elegant it looks, would be of little use if we had no better way to deal with the functional determinant (2.20) that arises in the integral (2.19). Luckily, we can make use of the fact, that a determinant of any Hermitian matrix $D^{ab}(x,y)$ can be expressed as a field integral [4]

$$\det D^{ab}(x,y) = \int \mathcal{D}c\,\mathcal{D}\bar{c}\exp\left(i\int d^4x\,d^4y\,\bar{c}^a(x)D^{ab}(x,y)c^b(y)\right), \tag{2.26}$$

where $c(x)$ and $\bar{c}(x)$ are *anticommuting* fields. In a similar manner, we can introduce anticommuting fields to our theory and rewrite the functional integral as

$$Z = \int \mathcal{D}\alpha \int \mathcal{D}c\,\mathcal{D}\bar{c}\,\mathcal{D}A\,\mathcal{D}\psi\, e^{i\int \mathcal{L}(A,\psi)+\mathcal{L}_{gf}(A,\psi;\xi)+\mathcal{L}_{gh}(c,\bar{c},A,\psi)}, \tag{2.27}$$

thus making the *effective* Lagrangian of our theory the sum of the usual plus gauge-fixing and ghost parts:

$$\mathcal{L}_{eff} = \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh}. \tag{2.28}$$

The ghost part is then

$$\mathcal{L}_{gh} = \bar{c}^a(x)\left(\frac{\delta G^a(A_\alpha,\psi_\alpha)}{\delta\alpha^b}\right)_{\alpha=0} c^b(x), \tag{2.29}$$

where a and b are indices in the adjoint representation of the gauge group.[1]

The reformulation of the functional determinant using noncommuting fields forces us to include fictitious particles called *ghost particles* (hence the name of the Lagrangian piece), two for each gauge field. When considering the Feynman diagrams of a process in a gauge-fixed theory, we must now include diagrams with ghost interactions too. Since the fields are noncommuting, the particles should be treated as fermions, yet they have spin 0, thus violating the spin-statistics theorem. This once again indicates that these particles are not physical and should be treated only as a technicality of the theory [5].

[1]They were omitted for clarity in the discussion of the Fadeev-Popov procedure.

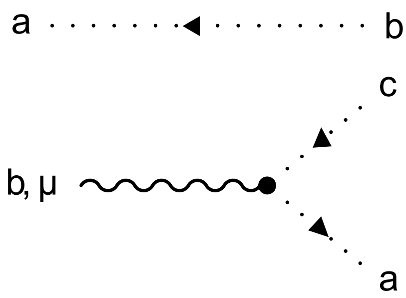

Figure 1: The ghost propagator and interaction vertex

Writing the ghost Lagrangian (2.29) out explicitly for a non-abelian gauge theory we get

$$\mathcal{L}_{gh} = \bar{c}^a \left(-\partial^2 \delta^{ac} - g\partial^\mu f^{abc} A^b_\mu\right) c^c, \tag{2.30}$$

and hence we see that the ghost propagator, represented as a dotted line in Feynman diagrams (Fig. 1), has a propagator given by

$$D^{ab}_{gh}(p) = \frac{i\delta^{ab}}{p^2} \tag{2.31}$$

and the ghost-gauge boson interaction vertex is given by

$$V^{abc}(p) = -gf^{abc}p^\mu. \tag{2.32}$$

Note that the vertex factor has the structure constants of the gauge group, meaning that it is non zero only for non-abelian gauge groups, i.e. abelian gauge theories, like QED, don't need to have ghosts.

2.2 Symmetry breaking

Suppose that $\phi(x)$ is a doublet of scalar fields, whose Lagrangian (the Klein-Gordon Lagrangian) is

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi). \tag{2.33}$$

Now suppose that the potential does not have a local minimum at $\phi_i = 0$, rather it has a continuous minimum at a certain length from the origin $|\phi| = \phi_0$ and a local maximum at $\phi_i = 0$. This is the Mexican hat potential, given by

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4. \tag{2.34}$$

Such a theory has an internal symmetry of $O(2)$, however now, instead of fluctuating around a zero value, the field will "roll down the hill" and acquire a non-zero vacuum expectation value. Thus we can write $\phi_1 = v + \kappa_1$ and $\phi_2 = \kappa_2$ with

$$v = \left(\frac{\mu^2}{\lambda}\right)^{\frac{1}{2}}. \tag{2.35}$$

Plugging the redefined fields into (2.33) and after doing some rearranging, the Lagrangian becomes

$$\mathcal{L}_{KG} = \frac{\mu^4}{4\lambda} + \frac{1}{2}\left((\partial_\mu \kappa_1)^2 + (\partial_\mu \kappa_2)^2\right) - \mu^2 \kappa_1^2 + O(\phi^3). \tag{2.36}$$

We see that the field κ_1 has a mass $\sqrt{2}\mu$ while κ_2 has no mass. Naturally, the Lagrangian has lost its internal symmetry of $O(2)$. We say that the original $O(2)$ symmetry has been *spontaneously broken* and a massless boson, called a *Nambu-Goldstone* boson has emerged.

This is a simple case of a more general *Goldstone theorem*, which states that whenever a continuous symmetry is spontaneously broken massless bosons emerge. These bosons correspond to the generators of the symmetry group that do not leave the vacuum invariant. In our example, any state along the gutter of the Mexican hat can be chosen to be the vacuum state, i.e. there is an $O(2)$ symmetry. However, by choosing a particular point in the gutter, we break this symmetry and thus we get one Nambu-Goldstone boson. It is easy to understand why it is massless — fluctuating along the gutter costs nothing in terms of energy, whereas the other boson has to "climb up a hill".

More interesting phenomena arise when we spontaneously break a scalar gauge theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi_i)^2 - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - V(\phi), \tag{2.37}$$

where $V(\phi_i)$ is the symmetry breaking potential defined in (2.34). The kinetic energy term when written out explicitly has the form

$$\frac{1}{2}(D_\mu \phi_i)^2 = \frac{1}{2}(\partial_\mu \phi_i)^2 + gA^a_\mu(\partial_\mu \phi_i T^a_{ij}\phi_j) + \frac{1}{2}g^2 A^a_\mu A^{b\mu}(T^a\phi)_i(T^b\phi)_i \tag{2.38}$$

Now if we let the fields acquire vacuum expectation values

$$\langle \phi_i \rangle = (\phi_0)_i \tag{2.39}$$

we see that the last term in (2.38) has the form of the gauge field mass term with the mass matrix

$$m_{ab} = g^2(T^a\phi_0)_i(T^b\phi_0)_i. \tag{2.40}$$

In the physical representation, where the mass matrix is diagonal, the diagonal elements m_{aa} are the masses of the gauge fields A^a_μ. If some generator T^a leaves the vacuum invariant, the corresponding gauge field remains massless.

This is basically the famous *Higgs mechanism*, where massless gauge fields acquire masses upon spontaneous symmetry breaking.

3 Renormalization

The need for a renormalization procedure arises when one considers higher order Feynman diagrams containing loops. In this section we will study the previously mentioned $U(1)$ gauge theory of quantum electrodynamics, i.e. QED, which turns out to be the unbroken $U(1)_{em}$ sector of the Standard Model, and see how divergences arise in simple higher order diagrams. We first discuss how they can be understood and treated. Later we develop a more systematic approach to the renormalization procedure.

3.1 Divergences in loop diagrams

Consider the simple loop diagram in Fig. 2, the electron self-energy diagram.

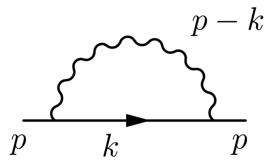

Figure 2: The electron self-energy diagram

The amplitude for it is given by

$$-i\Sigma^{[2]}(\not{p}) = (-ie)^2 \int \gamma^\nu \frac{-ig_{\mu\nu}}{k^2} \frac{i}{\not{p} - \not{k} - m} \gamma^\mu \frac{d^4k}{(2\pi)^4}, \tag{3.1}$$

where the superscript $^{[2]}$ indicates that this is a second order process. Looking at the integral it is immediately obvious that it is divergent, since there are three only powers of k in the denominator (even though it looks as a linear divergence, due to gauge invariance it is actually logarithmic [6]). At first sight this seems to be a disaster as we are calculating a correction to the tree level amplitude, but let's keep going. We define a *one-particle irreducible (1PI)* diagram as a diagram that cannot be split into two parts by removing a single line. The amplitude for the sum of all such diagrams is denoted by $-i\Sigma$, the absence of a superscript indicating that it is the 1PI electron self-energy with all orders included. To second order we have $\Sigma^{[2]} = \Sigma$. Any two-point diagram can then be expressed as a sum of such 1PI diagrams, as show in Fig. 3.

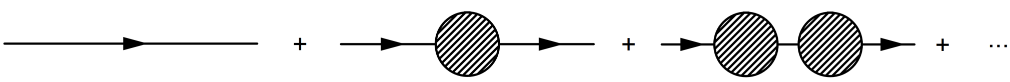

Figure 3: The full electron two-point function as a sum of 1PI diagrams

The total electron self-energy can now be written as a simple series

$$\frac{i(\not{p}+m_0)}{p^2-m_0^2}+\frac{i(\not{p}+m_0)}{p^2-m_0^2}(-i\Sigma(\not{p}))\frac{i(\not{p}+m_0)}{p^2-m_0^2}+\ldots. \tag{3.2}$$

By noting that Σ commutes with $\not{p}$ we can simplify the expression to

$$\frac{i}{\not{p}-m_0}+\frac{i}{\not{p}-m_0}\left(\frac{\Sigma(\not{p})}{\not{p}-m_0}\right)+\frac{i}{\not{p}-m_0}\left(\frac{\Sigma(\not{p})}{\not{p}-m_0}\right)^2+\ldots, \tag{3.3}$$

which we immediately recognize as a simple geometric series and can formally sum up. The resulting expression looks just like the electron propagator with a shifted electron mass, which makes sense, since it is unreasonable to think that quantum fluctuations will not shift it. Hence we introduce the electron propagator with full quantum corrections

$$S_F(p)=\frac{i}{\not{p}-m_0-\Sigma(\not{p})}. \tag{3.4}$$

It is now obvious that the Lagrangian parameter m_0 is *not* the physically measured electron mass, hence the subscript 0. The quantity m_0 is referred to as the *bare mass*. The physical mass is defined as the pole of the propagator and is given by the solution of the equation

$$m_p-m_0-\Sigma(m_p)=0. \tag{3.5}$$

We can now try to calculate the mass shift to second order:

$$\delta m=m_p-m_0=\Sigma^{[2]}(m_p)\approx\Sigma^{[2]}(m_0). \tag{3.6}$$

Calculating this shift requires performing the integral in (3.1), which is not an easy task. The first and most obvious problem is that it is divergent. The procedure of making it finite is called *regularization* and usually involves introducing some kind of momentum cut-off Λ. A notable example is the Pauli-Villars procedure which introduces massive particles to the theory. The cut-off can be justified by acknowledging the fact that out theory is not well defined for high momenta — we are parameterizing our ignorance. We hope that in the end, when sending the cut-off to infinity, a limit can be taken, which gives sensible results. With the integral regularized it is only a matter of skill to do it. The usual procedure is to introduce Feynman parameters to combine the denominators, Wick rotate the integration plane and do the simple resulting integral [4]. After performing the procedure and sending the cut-off to infinity the mass shift is

$$\lim_{\Lambda\to\infty}\delta m=\frac{3\alpha}{4\pi}\log\left(\frac{\Lambda^2}{m_0^2}\right). \tag{3.7}$$

A quantity which was supposed to be a small correction has turned out to be infinite.

Looking at the corrected electron propagator (3.4) we can see that not only is the pole shifted, but so is the residue. This shift can be interpreted as a rescaling of the field

$$\psi \to Z_2 \psi. \tag{3.8}$$

Close to the pole the denominator has the form

$$(\not{p} - m) \left(1 - \frac{d\Sigma}{d\not{p}}\Big|_{\not{p}=m}\right) + \mathcal{O}((\not{p} - m)^2) \tag{3.9}$$

and Z_2 is given by

$$Z_2 \approx 1 + \frac{d\Sigma}{d\not{p}}\Big|_{\not{p}=m}, \tag{3.10}$$

thus the second order shift of the field normalization constant is

$$\delta Z_2 = Z_2 - 1 = \frac{d\Sigma^{[2]}}{d\not{p}}\Big|_{\not{p}=m}. \tag{3.11}$$

This quantity also turns out to be divergent when the cut-off is taken to infinity.

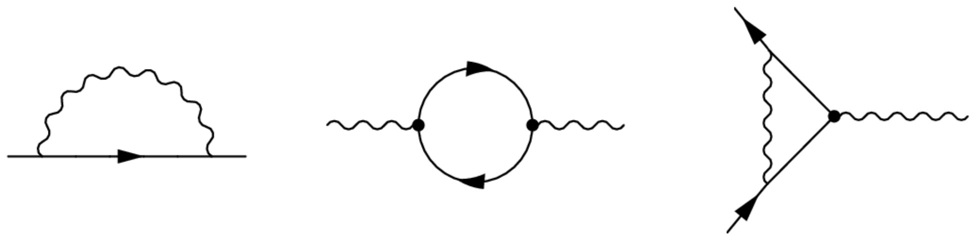

Figure 4: Divergent second order diagrams in QED

Further analysis of second order diagrams reveals that there are three different divergent diagrams in QED (c.f. [3]), all of which are shown in Fig. 4. The first diagram is the already familiar electron self-energy diagram. The second diagram is referred to as the *vacuum polarization diagram.* It is given by

$$i\Pi_2^{\mu\nu} \equiv (-ie)^2(-1) \int \frac{d^4k}{(2\pi)^4} \operatorname{tr} \left[\gamma^\mu \frac{i}{\not{k} - m} \gamma^\nu \frac{i}{\not{k} + \not{q} - m}\right]. \tag{3.12}$$

In analogy with the electron self-energy case, the sum of all 1PI diagrams of this kind is denoted as $\Pi^{\mu\nu}(p)$. The Ward identity restricts the tensor structure of this quantity so that we can factor out a projector[4]:

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2). \tag{3.13}$$

The same reasoning as with the electron self-energy leads to a corrected expression for the photon propagator

$$D_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2(1-\Pi(q^2))} + q_\mu q_\nu \text{ terms}, \tag{3.14}$$

where the second part is irrelevant for S-matrix elements, since it can be shown to cancel out using the Ward identity. Since $\Pi(q^2)$ has no poles at $q^2 = 0$, that implies that the photon remains massless to all orders in perturbation theory. The residue of the pole at $q^2 = 0$ is

$$Z_3 \equiv \frac{1}{1-\Pi(0)}. \tag{3.15}$$

Since this factor always enters twice for every interaction involving an exchange of a photon, it is reasonable to define it as a charge shift

$$e \to Z_3^{1/2} e. \tag{3.16}$$

This shift also turns out to be infinite.

The last divergent diagram in Fig. 4 is known as the *vertex correction diagram*. In general when the vertex triangle is replaced with the sum of all 1PI diagrams, the resulting amplitude is denoted as $\Gamma^\mu(p', p)$, where p' is the momentum of the outgoing electron and p is the momentum of the incoming electron (the photon momentum is determined by the conservation of momentum). Then to second order we have

$$\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu, \tag{3.17}$$

where $\delta\Gamma^\mu$ is the amplitude for the divergent second order diagram given in Fig. 4. Simple considerations of symmetry restrict the form of Γ^μ to

$$\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2), \tag{3.18}$$

where q is the momentum of the outgoing photon and $F_1(q^2)$ and $F_2(q^2)$ are called *form factors*. As expected, infinities arise when one tries to calculate the shifts in these form factors. While $\delta F_2(q^2)$ is actually finite, the divergence in $\delta F_1(q^2)$ pops out in the worst place possible — while evaluating $\delta F_1(0)$, which should be zero.

3.2 Resolving the divergences

The last subsection revealed that many divergent quantities arise in QED when one goes beyond tree-level calculations. The corrections, which should be small in principle, turn out to be infinite, which casts doubts on the theory as a whole. However, it turns out that if one carefully defines the quantities being calculated and expresses them in physically measurable quantities only, the result will always be finite — in that sense QED is a renormalizable theory.

When calculating the shift of the electron mass in one-loop level, we discovered that it is shifted by an infinite quantity. At first sight, this seems absurd, however, we deal with such infinities in classical physics all the time when we consider point sources, e.g. a point electric charge. In classical physics we are well aware that the concept of a point charge is somewhat artificial and that's why we tend to ignore the resulting infinities. However, since quantum field theory is more fundamental in that sense, it is reasonable to ask whether an infinite bare mass is acceptable. An analogy can be drawn to solid state physics, where a concept of an effective mass is introduced as a means to include various interaction effects in solids. The physical field mass is in that sense also an effective mass, which is a result of various self-interactions of the fields. The main difference is that in solid state physics it is possible to measure the bare electron mass and then calculate the effective mass from first principles. In field theory, on the other hand, we cannot step out of the physical vacuum — the bare mass is not a measurable quantity in principle, hence the fact that it is infinite is only a technicality. Once one defines all quantities only in physically measurable parameters, all the infinities drop out.

The fact that it is not possible to step outside of the vacuum, which is a very active environment with particles popping in and out of existence, forces one to carefully define the quantities being calculated. In free field theory, a propagator is defined as a correlation function

$$\langle 0 \mid T\psi(x)\bar{\psi}(y) \mid 0\rangle = \frac{i}{\not{p} - m_0}, \tag{3.19}$$

where $|0\rangle$ is the free field vacuum state. Obviously, the state of an interacting field vacuum is not the same as for a free field, hence the propagator should be defined as

$$\langle \Omega \mid T\psi(x)\bar{\psi}(y) \mid \Omega\rangle. \tag{3.20}$$

As we already discussed, the interactions of fields produce a mass shift. Since we still expect the propagator to be an orthogonal wave, the only other possible modification is introducing a scaling factor

$$S_F \approx \frac{iZ}{\not{p} - m}. \tag{3.21}$$

It's not hard to guess that Z is the same scaling factor we got in (3.8). This result is generalized by the *LSZ reduction formula*, which relates Feynman diagrams and S-matrix elements. It states that a matrix element for a particular process is the amplitude of the *amputated* (i.e. external legs have no self-interactions) Feynman diagram for the process times a square root of a scaling factor for each external fermion field. The theorem basically factors out the field self-interactions to scaling factors.

It is now clear that we did not consider the external field corrections when discussing the electron-photon interaction vertex amplitude (3.17). Applying the LSZ reduction formula we have

$$Z_2\Gamma^\mu = (1+\delta Z_2)(\gamma^\mu + \delta\Gamma^\mu) = \gamma^\mu + \delta\Gamma^\mu + \gamma^\mu \delta Z_2, \tag{3.22}$$

which in turn means that the first form factor shift has an extra term:

$$F_1(q^2) = 1 + \delta F_1(q^2) + \delta Z_2. \tag{3.23}$$

We stated that both shifts $\delta F_1(q^2)$ and δZ_2 are infinite in the previous subsection, but surprisingly enough, it turns out that their divergences both cancel each other, i.e.

$$\delta F_1(0) = -\delta Z_2, \tag{3.24}$$

hence

$$F_1(q^2) = 1 + [\delta F_1(q^2) - \delta F_1(0)] \tag{3.25}$$

is a finite and small correction to the first form factor.

The last divergence studied in the previous subsection is related to the rescaling of the elementary charge. We can dismiss the fact that the bare charge is infinite based on the same considerations of what is physically measurable and what is not. And since only the physical charge is measurable, we can always impose renormalization conditions for the charge, which will ensure that all physically measurable quantities turn out finite.

It is also worth mentioning that QED suffers not only from ultraviolet, but also *infrared divergences*[5]. These can also be treated by carefully defining the physical process being examined, e.g. one has to consider the emission of external photons of very large wavelength, that can't be detected because of the experimental threshold. Divergences of this type arise because of the fact that the photon is massless.

3.3 Renormalized perturbation theory

In the previous subsection we gave a detailed explanation for each of the divergences that arise in QED and ways to treat them. The way of dealing with divergences we adopted is usually referred to as *bare perturbation theory*. While the outlined procedures give a better view of the cancellations involved, most of the time it is a tedious task that requires much attention to details. A more practical approach is *renormalized perturbation theory*, which deals with physical quantities from the start and imposes *renormalization conditions* that assure that the result will turn out finite and consistent.

The procedure consists of a few simple steps. First we rescale the fields so as to absorb the field renormalizations:

$$\psi = Z_2^{-1/2}\psi_0, \tag{3.26}$$

$$A^\mu = Z_3^{-1/2} A_0^\mu, \tag{3.27}$$

where the fields with subscript 0 indicate the bare fields. The Lagrangian now becomes

$$\mathcal{L} = Z_2 \bar{\psi}(i\partial\!\!\!/ - m_0)\psi - e_0 Z_2 Z_3^{1/2} \bar{\psi}\gamma^\mu \psi A_\mu - \frac{1}{4} Z_3 (F^{\mu\nu})^2. \tag{3.28}$$

We can introduce another scaling factor for the electric charge

$$e_0 Z_2 Z_3^{1/2} = e Z_1, \tag{3.29}$$

where e is the physical charge measured at large distances, i.e. $q = 0$. Comparing this definition with (3.16) we see that they are consistent with each other only if $Z_2 = Z_1$. This actually turns out to be true as a case of the Ward identity and it suggests that the electron charge shift depends only on the renormalization of the photon field and is not a property of the electron. If that were not the case, introducing other charged particles to the theory would reveal that each particle has its own elementary charge, since they would be shifted differently.

By introducing another set of variables

$$\delta_1 = Z_1 - 1, \delta_2 = Z_2 - 1, \delta_3 = Z_3 - 1 \tag{3.30}$$

and

$$\delta_m = Z_2 m_0 - m, \tag{3.31}$$

we can rewrite the Lagrangian as

$$\begin{aligned} \mathcal{L} = &\bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}(F^{\mu\nu})^2 \\ &+ \bar{\psi}(i\delta_2\partial\!\!\!/ - \delta_m)\psi - e\delta_1\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}\delta_3(F^{\mu\nu})^2. \end{aligned} \tag{3.32}$$

We got rid of the bare parameters completely at the cost of introducing three new terms, called the counter terms, which absorb the infinite and unobservable shifts. These terms appear as three new types of interactions in Feynman diagrams as shown in Fig. 5.

Their amplitudes can be read off the Lagrangian, i.e. the electron propagator counter term is given by

$$i(p\!\!\!/\delta_2 - \delta_m), \tag{3.33}$$

the photon counter term is

$$-i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3 \tag{3.34}$$

and the electron-photon vertex counter term is

$$-ie\gamma^\mu\delta_1. \tag{3.35}$$

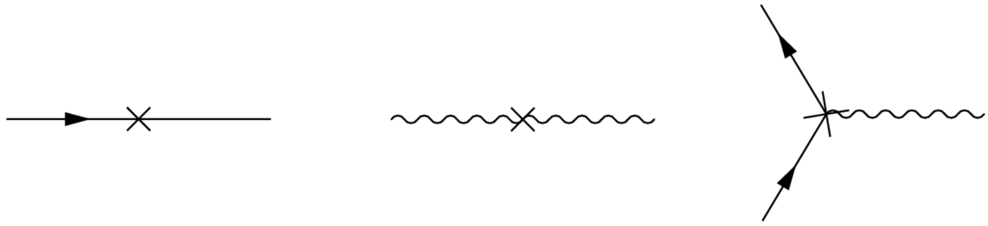

Figure 5: Counter term diagrams in QED

When drawing diagrams for a process one has to include the counter terms as well.

Last but not least we have to specify the renormalization conditions, which define the physical masses and electric charge and keep the field normalizations equal to 1. These are basically the self-consistency conditions discussed in the previous subsection, i.e. ensuring that the pole of the propagator is the physical mass etc. Since we are now discussing renormalized perturbation theory, the quantities $\Sigma(\not{p})$, $\Pi(q^2)$ and $\Gamma(p', p)$ are now defined to include the counter term diagrams as well as physical masses and charge. The renormalization conditions can then be stated as

$$\Sigma(\not{p} = m) = 0, \tag{3.36}$$

$$\frac{d}{d\not{p}}\Sigma(\not{p})\bigg|_{\not{p}=m} = 0, \tag{3.37}$$

$$\Pi(q^2 = 0) = 0, \tag{3.38}$$

$$-ie\Gamma^\mu(p' - p = 0) = -ie\gamma^\mu. \tag{3.39}$$

These relations can then be used to calculate the parameters δ_1, δ_2, δ_3 and δ_m in terms of the physical parameters at a specific order by doing the loop integrals explicitly. In the end the result will be finite and expressed in the physical parameters only.

3.4 Renormalization group

When discussing the photon self-energy diagram in the previous subsection we derived the corrected photon propagator (3.14), which we repeat here for convenience:

$$D_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2(1 - \Pi(q^2))} + q_\mu q_\nu \,\text{terms}\,. \tag{3.40}$$

This correction can be interpreted as a shift of the electron charge, which is the coupling constant, since a factor of e lies at each end of the photon propagator.

Thus we conclude that the parameter which was supposed to be a constant is actually a q dependent quantity

$$\alpha_0 \to \alpha_{eff}(q^2) = \frac{e_0^2/4\pi}{1 - \Pi(q^2)}. \tag{3.41}$$

Physically the q dependence of the electric charge can be explained by the effect of screening. In classical electrodynamics we are well aware of the fact that a medium can alter the strength of the electric field, which in turn can be interpreted as an alteration of the charge. This alteration is due to the screening effects of the charged particles in the medium. We expect a similar effect in the vacuum where electron-positron pairs can be created spontaneously and screen the electromagnetic interaction — an effect known as *vacuum polarization.* Far away from a test charge the vacuum is a dielectric medium, hence the strength of the interaction seems smaller. The interaction should increase as we near the test charge and enter the electron-positron cloud, since then we are able to see the real charge. Indeed, in one-loop level the integral for $\Pi_2(q^2)$ can be done in the limit $-q^2 \gg m^2$ yielding

$$\hat{\Pi}_2(q^2) = \frac{\alpha}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3} + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right], \tag{3.42}$$

where the hat indicates that the divergence is already subtracted. The effective coupling constant in this limit is

$$\alpha_{eff}(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\left(\frac{-q^2}{Am^2}\right)}, \tag{3.43}$$

where $A = \exp(5/3)$. As expected, the coupling gets stronger at small distances.

The fact that the coupling constant is not actually a constant is referred to as the *running of the coupling.* The proper tool for understanding this effect in particular and renormalization in general is the *renormalization group* approach. Since a detailed study of it is beyond the scope of this paper, we outline only the basic ideas here. The running of the couplings turns out to be closely related to the concept of scaling, which can be performed by scaling distance, i.e.

$$x \to xb, \tag{3.44}$$

or momenta

$$k \to k/b, \tag{3.45}$$

where b is the scaling factor. After performing a scaling operation, the initial Lagrangian of the theory can be brought back to its initial form by assuming that the parameters get shifted. In momentum space this corresponds to integrating out large momenta. By taking the limit $b = 1$ one can extract a continuous dependence of the parameters on the scale. This is exactly what we got in case of the QED coupling constant. Evolution of the coupling constants is described by the beta functions of the theory, defined as

$$\beta = \frac{M}{\delta M}\delta\lambda, \tag{3.46}$$

where M is the energy scale and λ is a coupling constant.

A detailed discussion about the renormalization group approach can be found in the references given at the end of the paper.

4 The Standard Model

After discussing the necessary tools we will now proceed and build the Standard Model. While some authors prefer a historic introduction, i.e. presenting and developing the theory from a historic perspective, we'll be taking an inductive approach — we will start by postulating the matter fields found in nature and their properties. From there on we will show how various phenomena emerge.

As stated before, mathematically, the Standard Model is a gauge field theory with the local gauge group

$$G_{loc} = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y. \tag{4.1}$$

Once the matter fields and their transformation laws are specified, the theory is essentially specified.

4.1 Construction

4.1.1 Matter fields

The matter fields in the Standard Model are all Dirac spinors with fixed helicity, meaning that The Standard Model is a *chiral* gauge theory. Since it is a relativistic theory, Lorentz invariance and causality enforce particle-antiparticle pairing. For chiral fields this means that a left-handed fermion at the same time describes a right-handed antifermion

$$\psi_L(x) \cong \psi_R^*(x). \tag{4.2}$$

Having that in mind, we can treat all fields as left-handed. The fields can then be classified into two families. The leptons are the electron, positron and the electron neutrino plus two equivalent generations — the muon and the tau generation. The hadron family is also composed of three generations, each consisting of up and down quarks and their antipartners, each of which can also have one of three colors — red, green or blue. The two families make a total of 45 particles, meaning that the free matter Lagrangian

$$\mathcal{L}_m = \sum_a \bar{\psi_{La}} i\gamma^\mu \partial_\mu \psi_{La} \tag{4.3}$$

has a global symmetry of $U(45)$.

Next we promote the subgroup (4.1) of the global symmetry group to be a local symmetry group, i.e. we gauge the theory. This implies that the fermion fields minimally couple to the gauge fields through the covariant derivative (2.10). The transformation laws of the matter fields are the simplest possible — they are either in the trivial or the fundamental representation of the subgroups. The representations are written out in table 1.

Table 1: Representations of the matter fields

subgroup	multiplets	representation
$SU(3)_C$	$\begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix}_L , \begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix}_R$	color and anticolor triplets
	leptons	color singlets
$SU(2)_L$	$\begin{pmatrix} \nu_l \\ l^- \end{pmatrix}_L , \begin{pmatrix} u \\ \tilde{d} \end{pmatrix}_L$	weak isospin doublets
	right handed fields	weak isospin singlets
$U(1)_Y$	all particles	weak hypercharge singlets

Here by q_c we denote any quark with the color c, by l^- we denote any lepton and the tilde on $\tilde{d}$ means that the down type (down, bottom or strange) quark is *Cabibbo-Kobayashi-Maskawa* rotated[5].

The quantum numbers of the fields under $SU(3)_C$ and $SU(2)_L$ transformations are determined by the group structures, however there are no commutation relations for the $U(1)_Y$ subgroup to fix it's normalization. Hence, in order for the correct quantum numbers to come out, a new generator is introduced

$$Q = T_3 + \frac{Y}{2}, \tag{4.4}$$

where T_3 is the third component of the weak isospin and Y is the weak hypercharge generator. The new generator Q will turn out to be the electromagnetic charge generator.

Now we can summarize all the weak quantum numbers in table 2.

Table 2: Weak quantum numbers for the matter fields

	Doublets				Singlets			
	$(\nu_l)_L$	$(l^-)_L$	$(u)_L$	$(\tilde{d})_L$	$(\nu_l)_R$	$(l^-)_R$	$(u)_R$	$(d)_R$
Q	0	-1	2/3	-1/3	0	-1	2/3	-1/3
T_3	1/2	-1/2	1/3	-1/2	0	0	0	0
Y	-1	-1	2/3	1/3	0	-2	4/3	-2/3

One might notice that we didn't mention the right-handed neutrino when enumerating the matter fields. From table 2 it is evident, that if we were to include it, it would not take part in any kind of interactions, since all it's quatum numbers are zero. One more reason for its exclusion is that it would allow for a non-zero neutrino mass as explained in the next subsections. During the early years of the Standard Model it was widely believed, that the neutrino was massles hence it was excluded. However, in the recent years, a small but non-vanishing neutrino mass *was* measured, forcing scientists to rethink the possibility of the right-handed neutrino's existence. Various mechanisms have been proposed to explain the incredibly small mass, e.g. the *seesaw mechanism.* However, due to the amount of overhead associated with it, the right-handed neutrino remains outside the Standard Model.

4.1.2 Gauge bosons and interactions

When talking about gauge theories we already saw from the fermion currents (2.15) that the fermions talk to each other only via spin 1 gauge bosons and the number of gauge bosons is the number of the gauge group generators. The gauge group of the Standard Model (4.1) has $8 + 3 + 1 = 12$ generators and thus 12 gauge bosons. The gauge fields are enumerated in table 3.

Table 3: The gauge fields of the Standard Model

subgroup	fields	coupling constant
$SU(3)_C$	$G_{\mu i}, i = 1, \ldots, 8$	g_s
$SU(2)_L$	$W_{\mu a}, a = 1, 2, 3$	g
$U(1)_Y$	B_μ	g'

All these fields should be massles, however only the $G_{\mu i}$ fields, identified with the eight gluons, and the photon are being observed — the other three gauge bosons appear to be massive. The next subsection will address this issue.

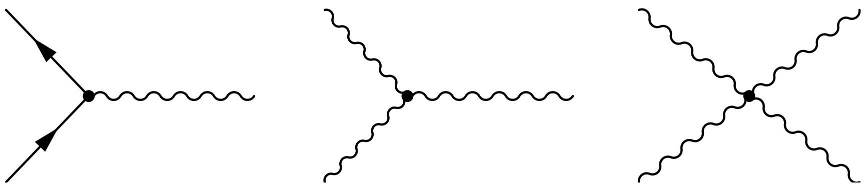

Figure 6: Fermion and gauge boson interaction vertices

Since the gauge group is a non-abelian group, gauge boson self interactions are also possible, i.e. the processes in Fig. 6 are allowed. Only the $U(1)$ gauge boson does not self-interact.

4.1.3 Electroweak unification

In order for the Standard Model to describe nature correctly, we have to generate masses for three of the $SU(2)_L \otimes U(1)_Y$ gauge bosons (the experimentally detected $W^\pm$ and Z^0 bosons) and one has to remain massles (the photon). Simply adding mass terms to the Lagrangian does not work, since the theory becomes non-renormalizable. The solution as suggested by Glashow, Salam and Weinberg is to utilize spontaneous symmetry breaking. We start by introducing an additional field doublet ϕ, called the *Higgs doublet*, which transforms as a spinor of $SU(2)_L$ and has a Lagrangian with a symmetry breaking form

$$\mathcal{L}_{Higgs} = (D_\mu\phi)^\dagger(D^\mu\phi) - \lambda(\phi^\dagger\phi)^2 + \mu^2(\phi^\dagger\phi). \tag{4.5}$$

We assign it a charge of 1/2 under the $U(1)_Y$ symmetry. Now let it acquire a vacuum expectation value

$$\langle\phi\rangle = \frac{1}{2}\begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{4.6}$$

A certain combination of the generators, namely the one given by (4.4), leaves this vacuum state invariant. This means that one gauge boson will remain massles while the other three will acquire masses. We calculate the new fields and their masses by sticking the fields given in table 3 and the vacuum expectation value (4.6) into the covariant derivative (2.10) and taking it's square. We get the mass terms

$$\Delta\mathcal{L} = \frac{1}{2}\frac{v^2}{4}\left(g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 + (-gA^3_\mu + g'B_\mu)^2\right). \tag{4.7}$$

The physical fields are the ones that are the eigenvalues of the mass and charge operators and are given as

$$W^\pm_\mu = \frac{1}{2}(A^1_\mu \mp iA^2_\mu) \tag{4.8}$$

with charges +1 and -1 and masses

$$m_W = g\frac{v}{2} \tag{4.9}$$

and

$$Z^0_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gA^3_\mu - g'B_\mu) \tag{4.10}$$

with zero charge and mass

$$m_Z = \sqrt{g^2+g'^2}\frac{v}{2}. \tag{4.11}$$

The masssles and zero charge field orthogonal to Z^0 is given by

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}(g'A^3_\mu + gB_\mu). \tag{4.12}$$

This is the already familiar electromagnetic vector potential.

The mass terms for the fermions are generated by introducing gauge invariant *Yukawa type couplings* between the lepton and quark Dirac fields and the scalar Higgs doublet. E.g. for the leptons

$$\mathcal{L}_{Yukawa,l} = -G_l\bar{l}_{eL}\phi e_R + h.c. \tag{4.13}$$

and after the Higgs doublet acquires the vacuum expectation value (4.6), this term gives a mass term for the electron

$$\mathcal{L}_{mass,l} = -\frac{1}{\sqrt{2}} G_l v \bar{e}_L e_R + h.c. \tag{4.14}$$

where the electron mass

$$m_e = \frac{1}{\sqrt{2}} G_l v \tag{4.15}$$

is expressed through the Yukawa coupling constant and the Higgs field vacuum expectation value. Note that since there is no right handed neutrino, we can't add a term in (4.13) that would couple it to the left-handed lepton doublet, hence generating a neutrino mass.

Quark masses are generated in the same way, by introducing two more (4.13) type Yukawa couplings

$$\mathcal{L}_{Yukawa,q} = -G_u \bar{l}_{qL} \phi d_R - G_u \bar{l}_{qL} \tilde{\phi} u_R + h.c. \tag{4.16}$$

generating masses for the u and d quarks. Here the tilde on $\tilde{\phi}$ denotes that the Higgs doublet is conjugated, this is done so that the net charge of the coupling under $SU(2)_L$ and $U(1)_Y$ would be zero. Since all three quark generations have identical weak quantum numbers, quarks from different generations can couple, meaning that the color interaction quark eigenstates are not exactly the same as the weak interaction eigenstates. They are related by the Cabbibo-Kobayashi-Maskawa rotation matrix.

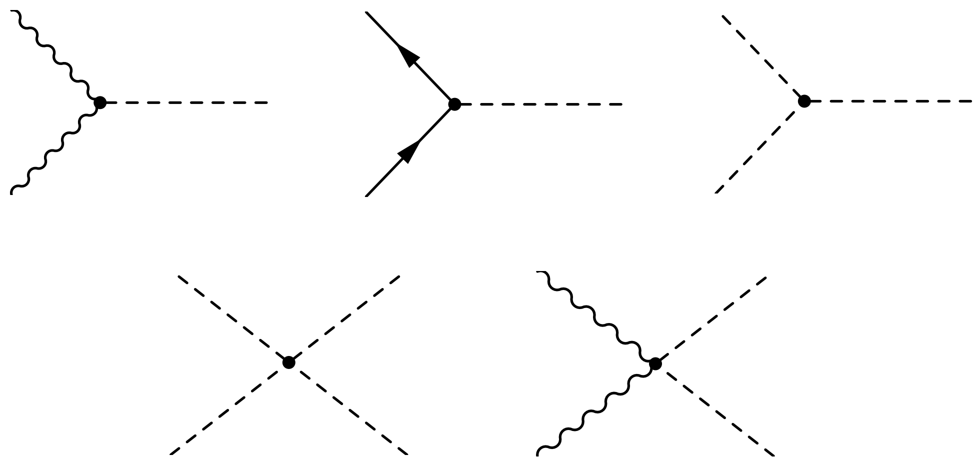

Figure 7: Higgs field couplings

4.1.4 The Higgs field

The introduced Higgs doublet ϕ has four degrees of freedom, three of which are "used up" by generating masses for the $W^{\pm}$ and Z^0 gauge fields, meaning we are left with a scalar field with one degree of freedom. This field has a vacuum expectation value v, thus in the unitary gauge we can write the Higgs doublet as

$$\phi(x) = \frac{1}{2} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{4.17}$$

where $h(x)$ is the *physical Higgs field*, meaning that $\langle h(x) \rangle = 0$. Quanta of the Higgs field are *Higgs bosons*, the only particles yet to be discovered in particle accelerators.

Plugging (4.17) into the Yukawa couplings (4.14) and (4.16) we get not only the fermion mass terms, but also see that the Higgs field couples to the fermions

$$\mathcal{L}_{Yukawa} = -\sum_f m_f \bar{\psi}_f \psi_f (1 + \frac{h}{v}) \tag{4.18}$$

with the coupling constant being proportional to the fermion mass. By plugging the same expression into the Higgs Lagrangian (4.5) and rearranging one can see that

$$\mathcal{L}_{Yukawa} + \mathcal{L}_{Higgs} = \mathcal{L}_{mass} + \mathcal{L}_h \tag{4.19}$$

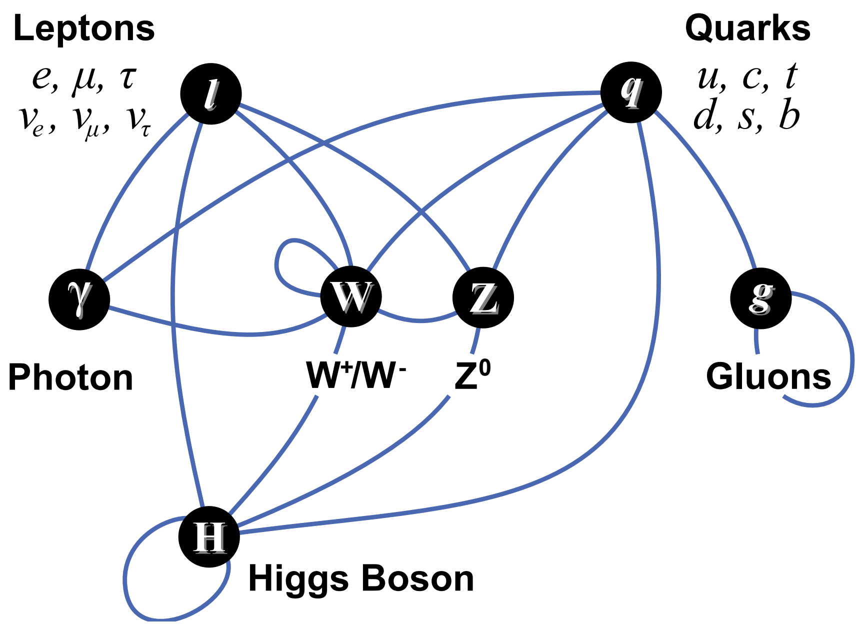

Figure 8: Particle interactions in the Standard Model

where $\mathcal{L}_{mass}$ are the fermion mass terms and

$$\begin{aligned}\mathcal{L}_h =& -\frac{1}{2}(\partial h)^2 - \frac{1}{2}m_h h^2 \\ &- \sum_f \frac{m_f}{v}\bar{\psi}_f \psi_f h + \frac{m_Z^2}{v} Z_\mu Z^\mu h + \frac{2m_W^2}{v} W_\mu^+ W^{-\mu} h \\ &- \frac{m_H^2}{2v} hhh - \frac{m_H^2}{8v^2} hhhh + \frac{m_W^2}{2v^2} W_\mu^+ W^{-\mu} hh + \frac{m_Z^2}{2v^2} Z_\mu Z^\mu hh\end{aligned} \tag{4.20}$$

is the Higgs field Lagrangian. Thus we see that the Higgs field has a mass and also couples to the massive gauge bosons and itself. All these processes are displayed in Fig. 7. All the particles and their interactions of the Standard Model are summarized in Fig. 8.

4.2 Gauge-fixing

As already discussed in the theoretical background section, in order to calculate something physical in a gauge theory, one has to first fix the gauge. In the functional integral formalism, i.e. when using Feynman rules, one has to apply the Fadeev-Popov procedure, which gets rid of all the physically equivalent field configurations in the path integral after the gauge has been fixed. Since gauge-fixing consists of picking constraining functions, which can be any functions of the fields, there are basically two types of gauge-fixing — linear and non-linear. The former is the standard choice most of the time, since it is a rather simple approach. We review it here first, followed by a discussion of non-linear gauge-fixing.

4.2.1 Linear gauge-fixing

The essence of the Fadeev-Popov procedure is to restrict the functional integral

$$Z = \int \mathcal{D}A\,\mathcal{D}\psi\, e^{i\int \mathcal{L}(A,\psi)} \tag{4.21}$$

by inserting a gauge-fixing constraint expressed as a delta function

$$\delta(G(x) - w(x)) \tag{4.22}$$

and to integrate over $w(x)$ with a Gaussian weight. Picking a constraining function $G(x)$ is then only a matter of convenience. Linear gauge-fixing by definition implies functions, which are linear in fields, e.g.

$$G(x) = \partial_\mu A^\mu(x). \tag{4.23}$$

After reviewing the Fadeev-Popov procedure it is now a straightforward procedure to calculate the gauge field propagator. Using functional methods we now get the two point correlation function from the effective Lagrangian:

$$\langle A_\mu(x)A_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+i\epsilon}\left(g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)e^{-ik(x-y)}, \tag{4.24}$$

leaving us with the gauge field propagator

$$D_{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{4.25}$$

This one-parameter class of gauge choices is known as the R_ξ *gauge*. Gauges with a specific value of ξ have their own names. Choices that are often convenient are [2]

$\xi = 0$	Landau gauge;
$\xi = 1$	Feynman-'t Hooft gauge;
$\xi \to \infty$	Unitary gauge.

When dealing with spontaneously broken gauge theories, the Standard Model in particular, the R_ξ gauge is introduced using the gauge-fixing functions defined as

$$G^a = \partial^\mu W^a_\mu - \xi_a m_a \chi^a, \tag{4.26}$$

which after integration results in gauge-fixing Lagrangian pieces

$$\mathcal{L}_{gf} = \sum \frac{1}{\xi_a}\left|G^a\right|^2, \tag{4.27}$$

where χ^a are the Goldstone bosons corresponding to the broken gauge symmetries and summing is performed in a manner which produces Lagrangian pieces which are invariant under the unbroken gauge symmetries of the theory. This choice corresponds to the massive gauge boson propagators

$$D_{\mu\nu}(k) = \frac{-i}{k^2-m_a^2+i\epsilon}\left(g_{\mu\nu}-(1-\xi_a)\frac{k_\mu k_\nu}{k^2-\xi_a m_a^2}\right) \tag{4.28}$$

and the effect of giving mass terms for the Goldstone bosons, which are proportional to the gauge parameter ξ_a, thus indicating that these particles, like the Fadeev-Popov ghosts, are unphysical.

Another possible gauge choice is the *unitary gauge*, which gets rid of the Goldstone bosons, resulting in the gauge field propagator [3]

$$D_{\mu\nu}(k) = \frac{-i}{k^2-m_a^2+i\epsilon}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{m_a^2}\right), \tag{4.29}$$

which can also be retrieved by sending $\xi_a \to \infty$ in the R_ξ gauge. Even though this gauge seems to be simpler, since we don't have to include Goldstone bosons in our diagrams, thus reducing the complexity of our expressions, it has it's drawbacks. In this case the gauge boson propagator does not fall off for $k^2 \to \infty$ producing problems in the high energy limit and thus, loop diagrams. This problem does not appear in the R_ξ gauge, which can even be used to prove the renormalizability of gauge theories in general.

4.2.2 Non-linear gauge-fixing

The constraining functions in the R_ξ gauge (4.26) are linear in fields, hence the whole procedure is called linear gauge-fixing. However, according to the Fadeev-Popov procedure, the constraining functions can be *any* functions of the fields. Of course, we should keep the dimension of the Lagrangian at most equal to 4 in order not to ruin renormalizability of the theory. By adding non-linear parts to the constraining functions we hope to get even more gauge-fixing parameters in our theory, which would be helpful in verifying the correctness of our results.

Up till now our discussion about gauge-fixing applied to general non-abelian gauge theories. From now on we will focus on the Standard Model and it's gauge fields. In order to avoid unnecessary complications, we should introduce constraining functions that are non-linear in fields and do not violate the unbroken gauge symmetries of the Standard Model, i.e. $U(1)_{em}$ gauge group of QED and the $SU(3)_c$ color group of QCD, meaning that the gauge-fixing Lagrangian terms should be neutral under these subgroups. As already discussed, the linear gauge-fixing Lagrangian part for the Standard Model is given by

$$\mathcal{L}_{gf} = -\frac{1}{\xi_W}G^-G^+ - \frac{1}{2\xi_Z}(G^Z)^2 - \frac{1}{2\xi_A}(G^A)^2 \tag{4.30}$$

and most of the time all the gauge-fixing parameters are made equal

$$\xi_W = \xi_Z = \xi_A \equiv \xi, \tag{4.31}$$

however we will not be taking this approach. The constraining functions are defined as

$$\begin{aligned} G^{\pm} &= -\frac{1}{\xi_W}\left(\partial_\mu W^{\pm\mu} - \xi_W m_W \chi^{\pm}\right), \\ G^Z &= -\frac{1}{\xi_Z}\left(\partial_\mu Z^{\mu} - \xi_Z m_Z \chi^{0}\right), \\ G^A &= -\frac{1}{\xi_A}\left(\partial_\mu A^{\mu}\right). \end{aligned} \tag{4.32}$$

We now introduce non-linear *additions* to the constraining functions in accordance to [8, 9]:

$$\begin{aligned} G^{\pm}_{nl} &= -ie\alpha\left(A_\mu W^{+\mu}\right) - ie\beta\frac{c_W}{s_W}\left(Z_\mu W^{+\mu}\right) - e\delta\frac{\xi_W}{2s_W}\left(h\chi^+\right) + ie\kappa\frac{\xi_W}{2s_W}\left(\chi^0\chi^+\right) \\ G^{Z}_{nl} &= -e\epsilon\frac{\xi_z}{2c_W s_W}\left(h\chi^0\right) \\ G^{A}_{nl} &= 0, \end{aligned} \tag{4.33}$$

where χ are the Goldstone bosons of the spontaneously broken gauge fields, h is the physical Higgs field and c_W and s_W are the cosine and sine of the Weinberg angle. The parameters α, β, δ, ϵ and κ are new gauge-fixing parameters which we are again free to choose. This choice of non-linear constraints also has a special feature that it does not change the quadratic part of the Lagrangian, i.e. the propagators are unaffected, only the vertices are affected.

4.2.3 Interaction vertices

We can now combine (4.32) and (4.33) and stick the resulting expression into the gauge-fixing Lagrangian piece (4.30) which we can then expand and collect terms which are cubic and quartic in fields (the interaction parts). As already mentioned, the quadratic parts which define the propagators are unaffected by non-linear gauge-fixing. After brute force expansion of the gauge-fixing Lagrangian we get these new pieces

$$\begin{aligned}
\mathcal{L}^{gf}_{vvvv} =& -\frac{\alpha^2e^2}{\xi_W}(W^-_\mu A^\mu W^+_\nu A^\nu) - \frac{\beta^2e^2c_W^2}{\xi_W s_W^2}(W^-_\mu Z^\mu W^+_\nu Z^\nu) \\
& -\frac{\alpha\beta e^2 c_W}{\xi_W s_W}(W^-_\mu A^\mu W^+_\nu Z^\nu + W^-_\mu Z^\mu W^+_\nu A^\nu),
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
\mathcal{L}^{gf}_{vvv} =& \frac{i\alpha e}{\xi_W}(W^+_\mu A^\mu \partial_\nu W^{-\nu} - W^-_\mu A^\mu \partial_\nu W^{+\nu}) \\
& + \frac{i\beta e c_W}{\xi_W s_W}(W^+_\mu Z^\mu \partial_\nu W^{-\nu} - W^-_\mu Z^\mu \partial_\nu W^{+\nu}),
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
\mathcal{L}^{gf}_{vvss} =& -\frac{i\alpha\delta e^2}{2s_W}(A_\mu W^{+\mu} h\chi^- - A_\mu W^{-\mu} h\chi^+) \\
& + \frac{\alpha\kappa e^2}{2s_W}(A_\mu W^{+\mu}\chi^0\chi^- + A_\mu W^{-\mu}\chi^0\chi^+) \\
& - \frac{i\beta\delta e^2 c_W}{2s_W^2}(Z_\mu W^{+\mu} h\chi^- - Z_\mu W^{-\mu} h\chi^+) \\
& + \frac{\beta\kappa e^2 c_W}{2s_W^2}(Z_\mu W^{+\mu}\chi^0\chi^- + Z_\mu W^{-\mu}\chi^0\chi^+),
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
\mathcal{L}^{gf}_{vvs} =& -i\alpha e m_W (W^+_\mu A^\mu \chi^- - W^-_\mu A^\mu \chi^+) \\
& - \frac{i\beta e m_W c_W}{s_W}(W^+_\mu Z^\mu \chi^- - W^-_\mu Z^\mu \chi^+),
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
\mathcal{L}^{gf}_{vss} =& \frac{\delta e}{2s_W}(h\chi^-\partial_\mu W^{+\mu} + h\chi^+\partial_\mu W^{-\mu}) \\
& + \frac{i\kappa e}{2s_W}(\chi^0\chi^-\partial_\mu W^{+\mu} - \chi^0\chi^+\partial_\mu W^{-\mu}) + \frac{\epsilon e}{2s_W c_W}(h\chi^0\partial_\mu Z^\mu)
\end{aligned} \tag{4.38}$$

$$\mathcal{L}^{gf}_{ssss} = -\frac{\delta^2 e^2\xi_W}{4s_W^2}(hh\chi^+\chi^-) - \frac{{}^2e^2\xi_W}{4s_W^2}(\chi^0\chi^0\chi^+\chi^-) - \frac{\epsilon^2e^2\xi_Z}{8s_W^2c_W^2}(hh\chi^0\chi^0) \tag{4.39}$$

$$\mathcal{L}^{gf}_{sss} = -\frac{\delta e\xi_W m_W}{s_W}(h\chi^+\chi^-) - \frac{\epsilon e\xi_Z m_Z}{2s_W c_W}(h\chi^0\chi^0). \tag{4.40}$$

Note that these are *additions* to the already present interaction terms in the full Lagrangian of the Standard Model. The introduced non-linear gauge-fixing Lagrangian term also affects the ghost interaction vertices. Looking at the definition of the ghost Lagrangian (2.29) we see that we will have new terms arising from the functional derivatives of the non-linear constraining functions (4.33) with respect to the gauge parameters. Writing everything out and leaving out the quadratic terms we get these interaction terms:

$$\begin{aligned}\mathcal{L}_{\alpha}^{ghi} =& + i\alpha e\bar{c}^{-}\left(A_{\mu}\partial^{\mu}c^{+} + W_{\mu}^{+}\partial^{\mu}c^{A}\right) - \alpha e^{2}\bar{c}^{-}\Big(A_{\mu}W^{+\mu}c^{A} + \tfrac{c_W}{s_W}A_{\mu}W^{+\mu}c^{Z} - \\ & -A_{\mu}A^{\mu}c^{+} - \tfrac{c_W}{s_W}A_{\mu}Z^{\mu}c^{+} - W_{\mu}^{+}W^{+\mu}c^{-} + W_{\mu}^{+}W^{-\mu}c^{+}\Big) + h.c.\end{aligned} \tag{4.41}$$

$$\begin{aligned}\mathcal{L}_{\beta}^{ghi} =& + i\beta e\tfrac{c_W}{s_W}\,\bar{c}^{-}\left(Z_{\mu}\partial^{\mu}c^{+} + W_{\mu}^{+}\partial^{\mu}c^{Z}\right) - \beta e^{2}\tfrac{c_W}{s_W}\,\bar{c}^{-}\Big(Z_{\mu}W^{+\mu}c^{A} + \tfrac{c_W}{s_W}Z_{\mu}W^{+\mu}c^{Z} - \\ & -Z_{\mu}A^{\mu}c^{+} - \frac{c_W}{s_W}Z_{\mu}Z^{\mu}c^{+} - \frac{c_W}{s_W}W_{\mu}^{+}W^{+\mu}c^{-} + \tfrac{c_W}{s_W}W_{\mu}^{+}W^{-\mu}c^{+}\Big) + h.c.\end{aligned} \tag{4.42}$$

$$\begin{aligned}\mathcal{L}_{\delta}^{ghi} =& - \delta\frac{e\xi_W m_W}{2s_W}\,\bar{c}^{-}hc^{+} + \delta\frac{e^{2}\xi_W}{4s_W^{2}}\,\bar{c}^{-}\Big(\chi^{+}\chi^{+}c^{-} + \chi^{+}\chi^{-}c^{+} + \\ & + \frac{1}{c_W}\chi^{+}\chi_{3}c^{Z} + i\frac{c_W^{2}-s_W^{2}}{c_W}\chi^{+}hc^{Z} + 2is_W\chi^{+}hc^{A} - i\chi_{3}hc^{+} - hhc^{+}\Big) + h.c.\end{aligned} \tag{4.43}$$

$$\begin{aligned}\mathcal{L}_{\kappa}^{ghi} =& i\kappa\frac{e\xi_W}{2s_W}\,\bar{c}^{-}\left(m_W\chi_{3}c^{+} + m_Z\chi^{+}c^{Z}\right) + \kappa\frac{e^{2}\xi_W}{4s_W^{2}}\,\bar{c}^{-}\Big(-\chi^{+}\chi^{+}c^{-} + \chi^{+}\chi^{-}c^{+} + \\ & + \frac{i}{c_W}\chi^{+}hc^{Z} + \frac{c_W^{2}-s_W^{2}}{c_W}\chi^{+}\chi_{3}c^{Z} + 2s_W\chi^{+}\chi_{3}c^{A} + i\chi_{3}hc^{+} - \chi_{3}\chi_{3}c^{+}\Big) + h.c.\end{aligned} \tag{4.44}$$

$$\begin{aligned}\mathcal{L}_{\varepsilon}^{ghi} =& - \epsilon\frac{e\xi_Z m_Z}{2s_W c_W}\bar{c}^{Z}hc^{Z} + \epsilon\frac{e^{2}\xi_Z}{4s_W^{2}c_W}\bar{c}^{Z}\Big(-i\chi^{+}hc^{-} + i\chi^{-}hc^{+} + \chi^{+}\chi^{0}c^{-} + \chi^{-}\chi^{0}c^{+} + \\ & + \frac{1}{c_W}\chi^{0}\chi^{0}c^{Z} - \frac{1}{c_W}hhc^{Z}\Big).\end{aligned} \tag{4.45}$$

Again, these are only additions to the already present interaction terms, which arise even without the non-linear parts of the constraining functions. However, this time we get two new types of interactions — ghost-ghost-vector-vector and ghost-ghost-scalar-scalar.

4.3 Renormalizing the Standard Model

In the previous subsection we outlined the major steps in renormalizing the theory of QED. In general, the whole renormalization procedure can be summed up to the following steps:

- Choose a set of independent parameters.
- Separate the bare parameters (and fields) into renormalized parameters (fields) and renormalization constants.
- Choose renormalization conditions to fix the counterterms.
- Express physical quantities in terms of the renormalized parameters.
- Choose input data in order to fix the value of the renormalized parameters.
- Evaluate predictions for physical quantities as functions of the input data.

The first three items specify the *renormalization scheme*, the most popular schemes being the *on-shell scheme*, which uses the fact that all external particles are physical, i.e. on-shell, as the boundary conditions, and the *minimal subtraction (MS) scheme*, which simply absorbs the divergent parts to the counterterms.

The set of independent parameters used for renormalizing the Standard Model usually is

$$e, M_W, M_Z, M_H, m_{f,i}, V_{ij} \tag{4.46}$$

and the renormalization conditions are very similar to the ones we used with QED, i.e. they are basically consistency conditions, specifying that things like particle masses are what we expect them to be.

The process of renormalizing the Standard Model, while rather simple in principle, is a very difficult and technical task — the sheer size of the Lagrangian of the Standard Model (given in appendix A) is overwhelming. Things are further complicated by the fact that the Standard Model includes spontaneous symmetry breaking and quark flavour mixing (which requires renormalizing the CKM matrix), hence requiring lots of attention to detail and manual labor. Therefore this topic is outside the scope of this paper — for a complete coverage of this topic see [2, 7].

5 Non-linear gauge-fixing in FeynArts

Calculating quantities using Feynman diagrams by hand is a tedious task, hence it is almost always done using some computer software package. One of the most popular packages is *FeynArts*[2], which runs on top of *Mathematica*, hence is available for a variety of platforms and operating systems [10].

5.1 About the package

FeynArts is actually only one part of the whole package, which also includes FormCalc and LoopTools. The basic structure of the calculation process is as follows: we first define the process we want to calculate by specifying the ingoing and outgoing particles. FeynArts is then responsible for finding all possible ways of connecting the incoming and outgoing particles, i.e. creating topologies of the soon to be Feynman diagrams. This includes counter terms and loop diagrams up to a specified level. Next, Feynman diagrams are generated by inserting appropriate particles into the topologies, which are allowed by the model being used. After the diagrams are completed, algebraic expressions are associated with the diagrams — this step is performed by FormCalc. Up till now, everything is being done in Mathematica. Algebraic simplification of the expressions usually requires a lot of computation power hence this step is being done internally in FormCalc by calling another program, called FORM, under the covers. Divergences are also taken care of during this step by calling various LoopTools subroutines where loop integrals are reduced to Passarino-Veltman integrals and the counter terms are evaluated using the defined renormalization scheme. In the end, FormCalc produces a FORTRAN program, which produces numerical results for the cross section given various process parameters, e.g. the center of mass energy of the process.

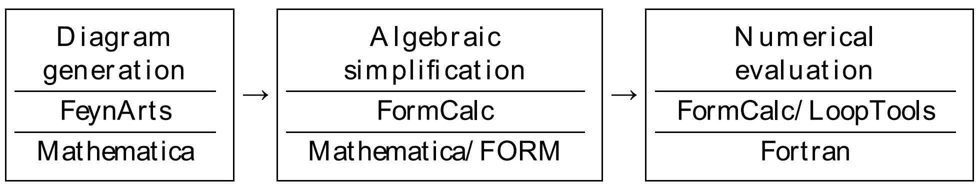

Figure 9: A schematic representation of FeynArts/FormCalc

One important feature of the package is that it is not fixed to some physical model, e.g. the Standard Model. Instead, it can be fully customized by providing *model files*. Feynman diagram generation is possible in three levels: generic

[2]http://www.feynarts.de/

fields, classes of fields and specific particles. The definition of the model is split into two pieces — *the generic model* and the *classes model file*. The generic model file basically defines the geometry of spacetime, what kind of particles do we have, i.e. scalars, spinors and vectors and what type of interactions are possible. FeynArts by default comes bundled with the `Lorentz.gen` file, which defines our Lorentz spacetime and the types of particles in it. It is used by many models, e.g. the Standard Model and the MSSM.

The model files define the particle content of the theory, give analytical expressions for the propagators, interaction vertices and define the renormalization scheme. Particle definitions are given in two levels: classes and particles, i.e. there is the fermion class containing the electron, quarks etc. Vertex definitions are closely related to the vertex *structure* definitions given in the generic model file. E.g. the `Lorentz.gen` generic model defines the structure of the vector-vector-vector-vector vertex as

$$C^{\mu\nu\rho\sigma}_{VVVV} = C^1_{VVVV} g^{\mu\nu} g^{\rho\sigma} + C^2_{VVVV} g^{\mu\rho} g^{\nu\sigma} + C^3_{VVVV} g^{\mu\sigma} g^{\nu\rho}, \tag{5.1}$$

leaving the coefficients unspecified. The `SM.mod` file defines these quantities for all possible vector-vector-vector-vector interactions, e.g. for four W^- boson scattering we have

```
C[ -V[3], -V[3], V[3], V[3] ] == I EL^2/SW^2 *
   { { 2, 4 dZe1 - 4 dSW1/SW + 4 dZW1  },
     {-1, -2 dZe1 + 2 dSW1/SW - 2 dZW1 },
     {-1, -2 dZe1 + 2 dSW1/SW - 2*dZW1 } }
```

Each line represents each of the constants C^n_{VVVV}, the first part of the line is the tree level multiplicative factor and the second part is the counterterm factor.

5.2 Renormalization in FeynArts

Renormalization in FeynArts is done almost automatically. All that needs to be specified are the counter term vertex expressions and the renormalization conditions. All of that can be specified in the model files. For example, the QED model uses a generic model file `QED.mod`, which defines the structure of a vector-vector interaction as

```
AnalyticalCoupling[ s1 V[i, mom1, {li1}],
                    s2 V[j, mom2, {li2}] ] ==
   G[1][s1 V[i], s2 V[j]] .
     { MetricTensor[li1, li2] ScalarProduct[mom1, mom2],
       MetricTensor[li1, li2],
       FourVector[mom1, li2] FourVector[mom2, li1] }
```

This means that a term for a vector-vector interaction in the most general case will be a sum of three factors proportional to $(p_1 \cdot p_2)g^{\mu\nu}$, $g^{\mu\nu}$ and $p_1^\mu p_2^\nu$. The model file then specifies these coefficients for specific vector particles at tree-level and one-loop level. So in QED we have

```
C[ V[1], V[1] ] == I * { {0, dZAA1}, {0, 0}, {0, -dZAA1} },
```

where the zeros in the first column indicate that there is no interaction of a photon with a photon in QED — that is simply the propagator and not an interaction. However we do have an interaction term in one-loop level. The constant `dZAA1` corresponds to the photon field renormalization constant δ_3, so when we combine the structure constants from the generic model file with the coefficients, the expression reduces to

$$-i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3, \tag{5.2}$$

which is exactly what we got in (3.34). All that's left is specifying the renormalization conditions, which are expressed in simple expressions like

```
RenConst[ dZAA1 ] := FieldRC[V[1]],
```

which identifies the constant `dZAA1` as the photon field renormalization constant. All of the rest is taken care by the package. The same ideas apply to other models, only the expressions can become much more complicated.

5.3 Implementation in FeynArts

The task of implementing non-linear gauge-fixing is now simple, we just have to go through all the interactions and add the new terms that arose due to non-linear gauge-fixing in (4.34) - (4.45). However, one difficulty arises when we try to implement the ghost interaction terms — non-linear gauge-fixing introduced new *types* of interactions, which are not present in `Lorentz.gen`, meaning that we have to modify it, too. Our implementation is based on a previous work on the subject [11], with one essential difference, that we chose not to update the existing model files, essentially producing new models, but to implement *updates* to the existing models. This way, we can use the updates with other models similar to the Standard Model. Furthermore, future updates of the `SM.mod` file won't affect this implementation. The update model files are available online[3].

Non-linear gauge-fixing is implemented in FeynArts as an addition to the `Lorentz.gen` generic model file and the `SM.m` model file for the Standard Model. The main purpose of this implementation is checking the correctness of the software package itself, since no physical result should depend on any of the gauge-fixing parameters and in the case of non-linear gauge-fixing we have 5 extra parameters.

[3]http://files.sauliaus.info/sm_nlg.tar.gz

5.4 Testing the implementation

The main goal of implementing non-linear gauge-fixing is to check whether various calculations performed using the FeynArts package are correct. Using non-linear gauge-fixing in particular is convenient, since it alters only interaction vertices, whereas linear gauge fixing alters the propagators, making the dependency on the linear gauge-fixing parameters too complicated to be used in practice. FeynArts uses the Feynman gauge implicitly, i.e. sets all linear gauge-fixing parameters to one, making any checks based on gauge-fixing impossible. We perform the checks by calculating cross sections for various processes in the Standard Model and checking whether any results depend on the non-linear gauge-fixing parameters, paying attention to not only tree level, but also one-loop level results.

5.4.1 Working examples

The first check is to determine whether the implementation itself is correct at all. The most obvious check is to see whether the new types of interactions appear in the list of diagrams produced for a particular process. We chose the process of an electron and a positron becoming top and anti-top quarks, or $e\bar{e} \to t\bar{t}$ in short as the main process for our checks at tree level. Looking at the diagrams we immediately recognize the new type of interactions introduced by non-linear gauge-fixing, e.g. vector-vector-ghost-ghost interactions. A few examples of such interactions are shown in Fig. 10.

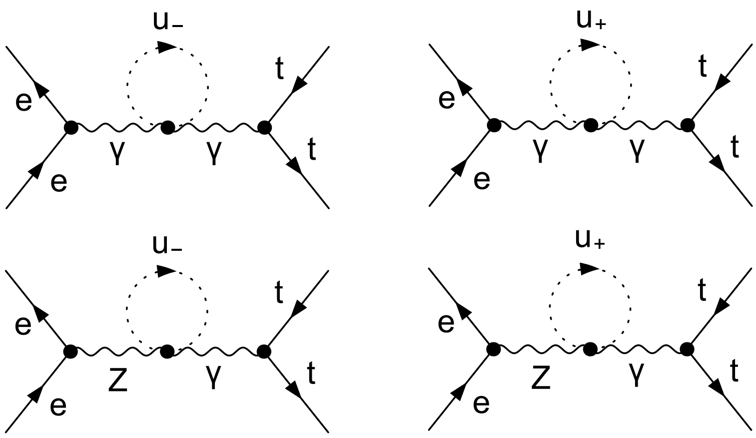

Figure 10: Vector-vector-ghost-ghost interaction examples

The next step is calculating a cross section for some specific set of initial conditions so that the result would be a number. We chose a simple scattering set up with the process energy set to $\sqrt{s} = 500\,\mathrm{GeV}$ and all particles being unpolarized. At first we tested the correctness of the results comparing them to the results of the same processes within the linearly gauge-fixed Standard

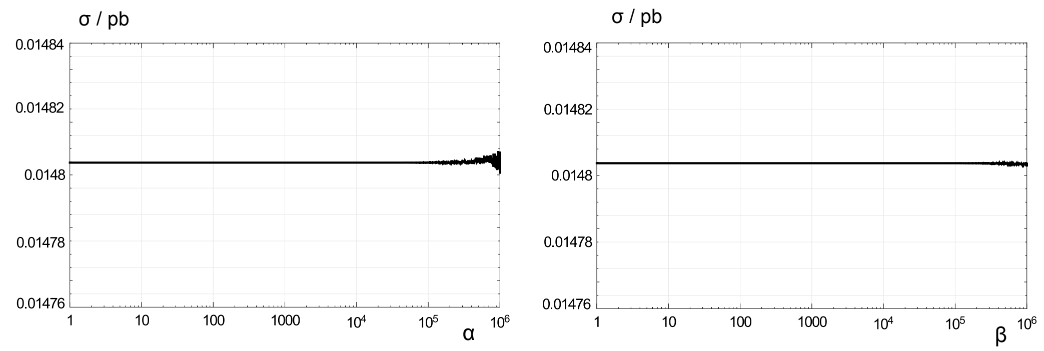

Figure 11: The dependence of the one-loop cross section correction for the $e\bar{e} \to t\bar{t}$ process at a specific energy and scattering angle on the α and β non-linear gauge-fixing parameters

Model. All the results matched with unlimited accuracy. This can be easily understood having in mind that FormCalc does a lot of algebraic simplifications. At tree level it is possible to cancel the gauge-fixing parameters completely, hence the results match exactly. That's why it is important to compare the results at one-loop level. In this case, when divergent quantities have to be regularized and treated, FormCalc is not able to carry out the cancelations algebraically and must rely on numerical methods. In this case some numerical fluctuations are to be expected. And that's exactly what we see when looking at the dependencies of a cross section for a specific process on the α and β parameters, defined in (4.33), in Fig. 11. We see that when the parameters reach a magnitude of 10^6, the change in the result is visible, yet numerically it is only $\approx 0.002\%$. Other working examples showing the same kind of dependencies include an electron-positron pair producing two photons, top and anti-top quarks producing two Higgs bosons, four W boson scattering, etc. — they all show only the negligible dependencies visible in Fig. 11.

5.4.2 Implementation limitations

Even though a lot of processes seem to be working correctly at one-loop level, we were able to find ones that do not, one example being an electron-positron pair producing a Z and Higgs boson pair. Even though the tree level cross sections show no dependence on the gauge-fixing parameters, at one-loop level, we see quadratic dependencies on the δ and ε parameters (Fig. 12). The problem seems to be related to massive vector bosons, the Z boson especially. All processes that fail seem to be showing the same kind of quadratic dependencies on the δ and ε parameters, e.g. the four Z boson scattering. The fact, that these dependencies are important, can be seen when comparing them with real dependencies of these cross sections on quantities like the process energy and the Higgs boson mass, seen in Fig. 13. Comparing them with the fluctuations

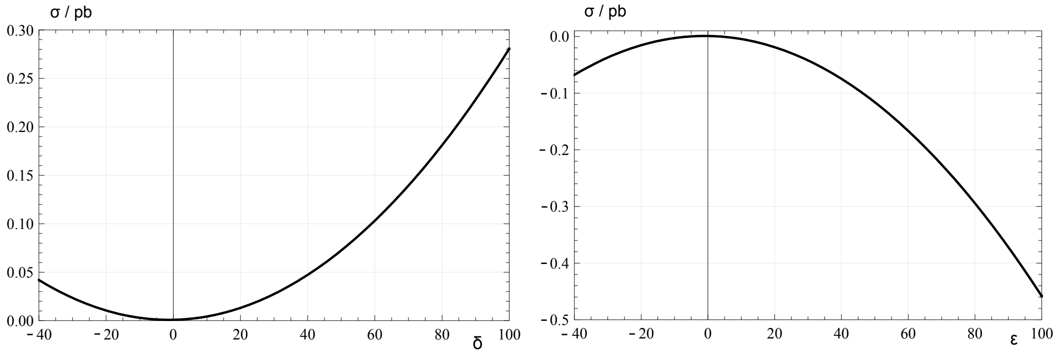

Figure 12: The dependence of the one-loop cross section correction for the $e\bar{e} \to ZH$ process at a specific energy and scattering angle on the δ and ε non-linear gauge-fixing parameters

visible in Fig. 11 also confirms our claim, that they are indeed neglibile.

Processes producing massive vector bosons are considered stable enough to be called physical, i.e. they can be detected in particle accelerators, however, on a theoretical level, they present some difficulties. Since FeynArts uses the Feynman gauge, the theory contains Goldstone bosons, which are massive — these bosons should be considered the longitudinal components of the corresponding massive vector bosons, meaning that processes with massive vector bosons should be considered in conjunction with analogous processes containing Goldstone bosons — they are incomplete when considered separately. This may be related to the discovered computational inconsistencies, however we were not able to define a process, be it with Goldstone bosons or discarding the longitudinal components at all, that would work correctly.

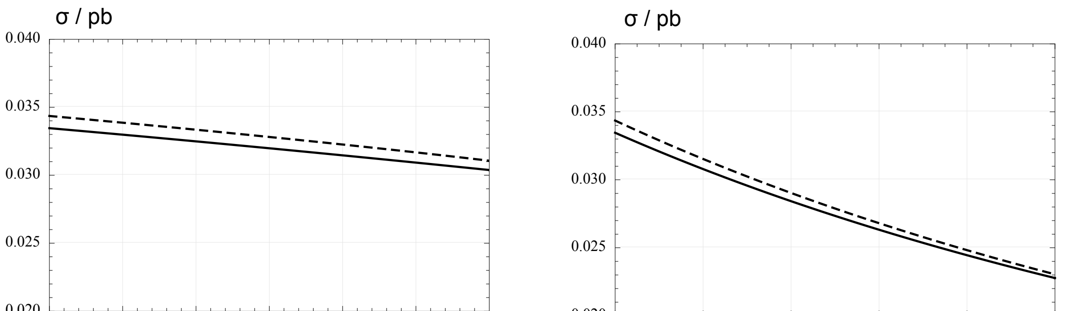

Figure 13: The dependencies of the cross sections for the $e\bar{e} \to ZH$ process on the mass of the Higgs boson m_H and the center of mass energy $\sqrt{s}$ of the process at tree level (solid line) and tree level including one-loop corrections (dashed line)

It is possible that FeynArts implements massive vector bosons incorrectly, however, we should not discard the possibility, that the implementation of non-linear gauge fixing is wrong or incomplete. Either way, since the problem is isolated to processes involving massive vector bosons, it is safe to say, that the implementation serves its purpose with more stable processes.

5.5 Gauge-fixing parameter renormalization

The possibility that the implementation of non-linear gauge-fixing is incomplete should also not be discarded. One obvious direction in extending it would be non-linear gauge-fixing parameter renormalization. Even though it is rather unusual, since the renormalization procedure usually fixes the physical parameters of the theory, theoretically it is possible to impose renormalization conditions for the non-linear gauge-fixing parameters, which should basically be conditions for the theory to be consistent at one-loop level. This process could resolve the encountered inconsistencies.

The topic of renormalizing the non-linear gauge-fixing parameters is beyond the scope of this work and should be the next major step at continuing it.

6 Conclusions

In this paper we presented an implementation of non-linear gauge-fixing in the Standard Model using the FeynArts/FormCalc package, by providing extensions to the already available Standard Model definitions. After reviewing the Standard Model in detail, including concepts like gauge-fixing and renormalization, we proceeded to checking the implementation of non-linear gauge-fixing at tree and one-loop levels.

The computational checks indicate, that the implementation is correct at tree level, yet has some limitations when working at one-loop level. We determined that we can't rely on it when calculating processes containing external massive vector bosons, since the results contain gauge parameter dependencies. The complications of spontaneous symmetry breaking and the appearance of Goldstone bosons seem to break the implementation. We concluded that a possible fix for this problem would include the renormalization of the gauge fixing parameters themselves with the renormalization conditions being simply the consistency conditions of the theory.

The main result of this work is the conclusion that the implementation can serve its purpose, i.e. check whether calculations carried out by FeynArts/ FormCalc are correct at tree and one-loop levels, as long as the process in question does not include external massive vector bosons. The encountered inconsistencies are also a major result, since the implementation served it's purpose — we found a possible bug in the FeynArts package or a missing piece in the implementation.

The next step in continuing this work would be solving the encountered inconsistencies, either by renormalizing the gauge fixing parameters or further investigating the ways that FeynArts handles massive vector bosons.

A Lagrangian of the Standard Model

In this appendix we give the full Lagrangian of the non-linearly fixed Standard Model grouped in terms based on species of the fields (vector, scalar, fermion, ghost) involved.

$$
\begin{aligned}
\mathcal{L} =& \mathcal{L}_v + \mathcal{L}_s + \mathcal{L}_f + \mathcal{L}_g + \\
&+ \mathcal{L}_{vvvv} + \mathcal{L}_{vvv} + \mathcal{L}_{vvss} + \mathcal{L}_{vvs} + \mathcal{L}_{vss} + \mathcal{L}_{ssss} + \mathcal{L}_{sss} + \\
&+ \mathcal{L}_{ffv} + \mathcal{L}_{ffs} + \\
&+ \mathcal{L}_{ggv} + \mathcal{L}_{ggs} + \mathcal{L}_{ggvv} + \mathcal{L}_{ggss}.
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\mathcal{L}_v =& - W^-_\mu(-g^{\mu\nu}\partial^2 + (1-\xi_W)\partial^\mu\partial^\nu - m_W^2 g^{\mu\nu})W^+_\nu \\
&- \frac{1}{2}Z_\mu(-g^{\mu\nu}\partial^2 + (1-\xi_Z)\partial^\mu\partial^\nu - m_Z^2 g^{\mu\nu})Z_\nu - \frac{1}{2}A_\mu(-g^{\mu\nu}\partial^2 + (1-\xi_A)\partial^\mu\partial^\nu)A_\nu \\
\mathcal{L}_s =& \chi^-(-\partial^2 - \xi_W m_W^2)\chi^+ + \frac{1}{2}\chi_3(-\partial^2 - \xi_Z m_Z^2)\chi_3 + \frac{1}{2}h(-\partial^2 - m_H^2)h \\
\mathcal{L}_g =& \bar{c}^-(-\partial^2 - \xi_W m_W^2)c^+ + \bar{c}^+(-\partial^2 - \xi_W m_W^2)c^- + \bar{c}^Z(-\partial^2 - \xi_W m_W^2)c^Z + \bar{c}^A(-\partial^2)c^A \\
\mathcal{L}_f =& \sum_{j=1}^{3} \bar{l}_j(i\partial\!\!\!/ - m_{l,j})l_j + \bar{\nu_j}(i\partial\!\!\!/)\nu_j + \bar{u_j}(i\partial\!\!\!/ - m_{u,j})u_j + \bar{d}_j(i\partial\!\!\!/ - m_{d,j})d_j
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
\mathcal{L}_{vvvv} = -\frac{e^2}{s_W^2} &\left\{ \tfrac{1}{2}\left(W^-_\mu W^{+\mu} W^-_\nu W^{+\nu} - W^-_\mu W^{-\mu} W^+_\nu W^{+\nu}\right)\right. \\
&+ c_W^2\left(W^-_\mu W^{+\mu} Z_\nu Z^\nu - (1-\beta^2/\xi_W) W^-_\mu Z^\mu W^+_\nu Z^\nu\right) \\
&+ s_W^2\left(W^-_\mu W^{+\mu} A_\nu A^\nu - (1-\alpha^2/\xi_W) W^-_\mu A^\mu W^+_\nu A^\nu\right) \\
&\left. + s_W c_W\left(2W^-_\mu W^{+\mu} A_\nu Z^\nu - (1-\alpha\beta/\xi_W)(W^-_\mu A^\mu W^+_\nu Z^\nu + W^-_\mu Z^\mu W^+_\nu A^\nu)\right)\right\}.
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\mathcal{L}_{vvv} =& - ie\frac{c_W}{s_W}\left[Z^\nu\left(W^{-\mu}\partial_\nu W^+_\mu - W^{+\mu}\partial_\nu W^-_\mu\right) + W^{+\nu}\left(Z^\mu\partial_\nu W^-_\mu - W^{-\mu}\partial_\nu Z_\mu\right)\right. \\
&\left. + W^{-\nu}\left(W^{+\mu}\partial_\nu Z_\mu - Z^\mu\partial_\nu W^+_\mu\right) - \beta/\xi_W(W^+_\mu Z^\mu\partial_\nu W^{-\nu} - W^-_\mu Z^\mu \partial_\nu W^{+\nu})\right] \\
&- ie\left[A^\nu\left(W^{-\mu}\partial_\nu W^+_\mu - W^{+\mu}\partial_\nu W^-_\mu\right) + W^{+\nu}\left(A^\mu\partial_\nu W^-_\mu - W^{-\mu}\partial_\nu A_\mu\right)\right. \\
&\left. + W^{-\nu}\left(W^{+\mu}\partial_\nu A_\mu - A^\mu\partial_\nu W^+_\mu\right) - \alpha/\xi_W(W^+_\mu A^\mu\partial_\nu W^{-\nu} - W^-_\mu A^\mu\partial_\nu W^{+\nu})\right]
\end{aligned}
\tag{A.4}
$$

$$\begin{aligned}\mathcal{L}_{vvss} =& \frac{e^2}{2s_W}\big(i(1-\alpha\delta)(A_\mu W^{+\mu}h\chi^- - A_\mu W^{-\mu}h\chi^+) \\ &\qquad\qquad -(1-\alpha\kappa)(A_\mu W^{+\mu}\chi_3\chi^- + A_\mu W^{-\mu}\chi_3\chi^+)\big) \\ &+ \frac{e^2}{2c_W}\big(-i(1+\beta\delta c_W^2/s_W^2)(Z_\mu W^{+\mu}h\chi^- - Z_\mu W^{-\mu}h\chi^+) \\ &\qquad\qquad +(1+\beta\kappa c_W^2/s_W^2)(Z_\mu W^{+\mu}\chi_3\chi^- + Z_\mu W^{-\mu}\chi_3\chi^+)\big) \\ &+ e^2 A_\mu A^\mu\chi^+\chi^- + \frac{e^2(c_W^2-s_W^2)}{s_W c_W}A_\mu Z^\mu\chi^+\chi^- + \frac{e^2(c_W^2-s_W^2)^2}{4s_W^2c_W^2}Z_\mu Z^\mu\chi^+\chi^- \\ &+ \frac{e^2}{4s_W^2}\big(W_\mu^+W^{-\mu}hh + W_\mu^+W^{-\mu}\chi_3\chi_3 + 2W_\mu^+W^{-\mu}\chi^+\chi^-\big) \\ &+ \frac{e^2}{8s_W^2c_W^2}\left(Z_\mu Z^\mu hh + Z_\mu Z^\mu\chi_3\chi_3\right).\end{aligned} \tag{A.5}$$

$$\begin{aligned}\mathcal{L}_{vvs} =& em_W(1-\alpha)\big(iW_\mu^+A^\mu\chi^- - iW_\mu^-A^\mu\chi^+\big) \\ &+ \frac{em_W s_W}{c_W}(1+\beta c_W^2/s_W^2)\big(-iW_\mu^+Z^\mu\chi^- + iW_\mu^-Z^\mu\chi^+\big) \\ &+ \frac{em_W}{s_W}W_\mu^+W^{-\mu}h + \frac{em_W}{2s_Wc_W^2}Z_\mu Z^\mu h\end{aligned} \tag{A.6}$$

$$\begin{aligned}\mathcal{L}_{vss} =& \frac{e}{2s_W}\big[W^{+\mu}(h\partial_\mu\chi^- - \chi^-\partial_\mu h) + \delta(h\chi^-\partial_\mu W^{+\mu}) \\ &\qquad\qquad +W^{-\mu}(h\partial_\mu\chi^+ - \chi^+\partial_\mu h) + \delta(h\chi^+\partial_\mu W^{-\mu})\big] \\ &+\frac{ie}{2s_W}\big[W^{+\mu}(\chi_3\partial_\mu\chi^- - \chi^-\partial_\mu\chi_3) + \kappa(\chi_3\chi^-\partial_\mu W^{+\mu}) \\ &\qquad\qquad -W^{-\mu}(\chi_3\partial_\mu\chi^+ - \chi^+\partial_\mu\chi_3) - \kappa(\chi_3\chi^+\partial_\mu W^{-\mu})\big] \\ &+ieA^\mu(\chi^-\partial_\mu\chi^+ - \chi^+\partial_\mu\chi^-) + ie\frac{c_W^2-s_W^2}{2s_Wc_W}Z^\mu(\chi^-\partial_\mu\chi^+ - \chi^+\partial_\mu\chi^-) \\ &+\frac{e}{2s_Wc_W}\left[Z^\mu(h\partial_\mu\chi_3 - \chi_3\partial_\mu h) + \varepsilon(h\chi_3\partial_\mu Z^\mu)\right].\end{aligned} \tag{A.7}$$

$$\begin{aligned}\mathcal{L}_{ssss} =& -\frac{e^2m_H^2}{32s_W^2m_W^2}\big(hhhh + \chi_3\chi_3\chi_3\chi_3 + 4\chi^+\chi^+\chi^-\chi^- \\ &\quad +2(1+2\varepsilon^2\xi_Z m_Z^2/m_H^2)hh\chi_3\chi_3 + 4(1+2\delta^2\xi_W m_W^2/m_H^2)hh\chi^+\chi^- \\ &\quad +4(1+2\kappa^2\xi_W m_W^2/m_H^2)\chi^+\chi^-\chi_3\chi_3\big)\end{aligned} \tag{A.8}$$

$$\mathcal{L}_{sss} = -\frac{em_H^2}{4s_Wm_W}\big(hhh + (1+2\varepsilon\xi_Z m_Z^2/m_H^2)h\chi_3\chi_3 + 2(1+2\delta\xi_W m_W^2/m_H^2)h\chi^+\chi^-\big). \tag{A.9}$$

$$\mathcal{L}_{ffv} = \sum_{j=1}^{3} \left\{ \frac{e}{\sqrt{2}s_W} W_\mu^+ (\bar{\nu_j}\gamma^\mu l_{L,j} + \bar{u}_{L,j}\gamma^\mu d_{L,j}) + \frac{e}{\sqrt{2}s_W} W_\mu^- (\bar{l}_{L,j}\gamma^\mu \nu_j + \bar{d}_{L,j}\gamma^\mu u_{L,j}) \right. \tag{A.10}$$

$$+ \frac{e}{s_W c_W} Z_\mu \left[+\tfrac{1}{2}\bar{\nu_j}\gamma^\mu \nu_j - \tfrac{1}{2}\bar{l}_{L,j}\gamma^\mu l_{L,j} + \tfrac{1}{2}\bar{u}_{L,j}\gamma^\mu u_{L,j} - \tfrac{1}{2}\bar{d}_{L,j}\gamma^\mu d_{L,j} \right.$$

$$\left. + \sin^2\theta_w \bar{l_{L,j}}\gamma^\mu l_{L,j} - \tfrac{2}{3}\sin^2\theta_w \bar{u_j}\gamma^\mu u_j + \tfrac{1}{3}\sin^2\theta_w \bar{d_j}\gamma^\mu d_j \right]$$

$$\left. + eA_\mu \left(-\bar{l_{L,j}}\gamma^\mu l_{L,j} + \tfrac{2}{3}\bar{u_j}\gamma^\mu u_j - \tfrac{1}{3}\bar{d_j}\gamma^\mu d_j \right) \right\}$$

$$\mathcal{L}_{ffs} = \sum_{j=1}^{3} \left\{ \frac{ie}{\sqrt{2}s_W m_W} \chi^+ (m_{l,j}\bar{\nu}_{L,j} l_{R,j} + m_{d,j}\bar{u}_{L,j} d_{R,j} - m_{u,j}\bar{u}_{R,j} d_{L,j}) \right. \tag{A.11}$$

$$+ \frac{ie}{\sqrt{2}s_W m_W} \chi^- (-m_{l,j}\bar{l}_{R,j}\nu_{L,j} + m_{u,j}\bar{d}_{L,j} u_{R,j} - m_d \bar{d}_{R,j} u_{L,j})$$

$$- \frac{ie}{2s_W m_W} \chi_3 (m_{l,j}\bar{l}_j\gamma^5 l_j + m_{d,j}\bar{d}_j\gamma^5 d_j - m_{u,j}\bar{u_j}\gamma^5 u_j)$$

$$\left. - \frac{e}{2s_W m_W} (m_{l,j}\bar{l}_j e_j + m_{d,j}\bar{d}_j d_j + m_{u,j}\bar{u_j} u_j) \right\}$$

$$\mathcal{L}_{ggv} = + ieW^{+\mu} \left((\partial_\mu \bar{c}^-) c^A + \alpha \bar{c}^- \partial_\mu c^A \right) + ie\frac{c_W}{s_W} W^{+\mu} \left((\partial_\mu \bar{c}^-) c^Z + \beta \bar{c}^- \partial_\mu c^Z \right) \tag{A.12}$$

$$- ieA^\mu \left((\partial_\mu \bar{c}^-) c^+ - \alpha \bar{c}^- \partial_\mu c^+ \right) - ie\frac{c_W}{s_W} Z^\mu \left((\partial_\mu \bar{c}^-) c^+ - \beta \bar{c}^- \partial_\mu c^+ \right)$$

$$- ieW^{-\mu} \left((\partial_\mu \bar{c}^+) c^A + \alpha \bar{c}^+ \partial_\mu c^A \right) - ie\frac{c_W}{s_W} W^{-\mu} \left((\partial_\mu \bar{c}^+) c^Z + \beta \bar{c}^+ \partial_\mu c^Z \right)$$

$$+ ieA^\mu \left((\partial_\mu \bar{c}^+) c^- - \alpha \bar{c}^+ \partial_\mu c^- \right) + ie\frac{c_W}{s_W} Z^\mu \left((\partial_\mu \bar{c}^+) c^- - \beta \bar{c}^+ \partial_\mu c^- \right)$$

$$- ie\frac{c_W}{s_W} (\partial_\mu \bar{c}^Z) \left(W^{+\mu} c^- - W^{-\mu} c^+ \right)$$

$$- ie(\partial_\mu \bar{c}^A) \left(W^{+\mu} c^- - W^{-\mu} c^+ \right)$$

$$\mathcal{L}_{ggs} = + \frac{ie\xi_W m_W}{2s_W} \bar{c}^- \left(\frac{c_W^2 - s_W^2 + \kappa}{c_W} \chi^+ c^Z + 2s_W \chi^+ c^A - (1-\kappa)\chi_3 c^+ + i(1+\delta) h c^+ \right)$$

$$- \frac{ie\xi_W m_W}{2s_W} \bar{c}^+ \left(\frac{c_W^2 - s_W^2 + \kappa}{c_W} \chi^- c^Z + 2s_W \chi^- c^A - (1-\kappa)\chi_3 c^- - i(1+\delta) h c^- \right)$$

$$- \frac{ie\xi_Z m_Z}{2s_W} \bar{c}^Z \left(\chi^+ c^- - \chi^- c^+ - \frac{i}{c_W}(1+\varepsilon) h c^Z \right). \tag{A.13}$$

$$\mathcal{L}_{ggvv} = - e^2 \bar{c}^- \left(\alpha A_\mu W^{+\mu} c^A + \alpha \tfrac{c_W}{s_W} A_\mu W^{+\mu} c^Z + \beta \tfrac{c_W}{s_W} Z_\mu W^{+\mu} c^A + \beta \tfrac{c_W^2}{s_W^2} Z_\mu W^{+\mu} c^Z - \right.$$

$$- \alpha A_\mu A^\mu c^+ - \beta \tfrac{c_W^2}{s_W^2} Z_\mu Z^\mu c^+ - (\alpha+\beta) \tfrac{c_W}{s_W} A_\mu Z^\mu c^+ -$$

$$\left. -(\alpha + \beta \tfrac{c_W^2}{s_W^2}) W_\mu^+ W^{+\mu} c^- + (\alpha + \beta \tfrac{c_W^2}{s_W^2}) W_\mu^+ W^{-\mu} c^+ \right) + \text{h.c.} \tag{A.14}$$

$$
\begin{aligned}
\mathcal{L}_{ggss} = \Big[&\frac{e^2\xi_W}{4s_W^2}\,\bar{c}^-\,\big((\delta-\kappa)\chi^+\chi^+c^- + (\delta+\kappa)\chi^+\chi^-c^+ + \\
&+\frac{\delta+\kappa(c_W^2-s_W^2)}{c_W}\chi^+\chi_3c^Z + i\frac{\delta(c_W^2-s_W^2)+\kappa}{c_W}\chi^+hc^Z + i(\kappa-\delta)\chi_3hc^+ + \\
&+2i\delta s_W\chi^+hc^A + 2\kappa s_W\chi^+\chi_3c^A - \delta hhc^+ - \kappa\chi_3\chi_3c^+\big) + \text{h.c.}\Big] \\
&+\varepsilon\frac{e^2\xi_Z}{4s_W^2c_W}\bar{c}^Z\left(-i\chi^+hc^- + i\chi^-hc^+ + \chi^+\chi_3c^- + \chi^-\chi_3c^+ + \frac{1}{c_W}\chi_3\chi_3c^Z - \frac{1}{c_W}hhc^Z\right).
\end{aligned}
\tag{A.15}
$$

B FeynArts model file listings

This appendix contains the Mathematica code for the implementation of non-linear gauge-fixing in the Standard Model using FeynArts. The implementation consists of two files: the generic Lorentz model file extension and the Standard Model extension.

The file `Lorentz_nlg.gen` specifies the additional types of interaction vertices that appear because of non-linear gauge fixing:

```
GenericCouplingTypeQ[c_, type_] :=
(Map[Head, (c /. {lhs_ == rhs_ -> lhs})[[All, 2]]]
/. AnalyticalCoupling -> List) == type

GenericCouplingIndex[type_] :=
 Position[M$GenericCouplings,
    Select[M$GenericCouplings,
      GenericCouplingTypeQ[#, type] &][[1]]][[1, 1]]

AppendGenericCoupling[type_, item_] :=
 Module[{idx, orig, lhs, rhs},
  idx = GenericCouplingIndex[type];
  orig = M$GenericCouplings[[idx]];
  rhs = orig /. lhs_ == rhs_ -> rhs;
  lhs = orig /. lhs_ == rhs_ -> lhs;
  rhs = Insert[rhs, item, {2, Length[rhs[[2]]] + 1}];
  M$GenericCouplings[[idx]] = Equal[lhs, rhs];
  ]

AppendGenericCoupling[{V, V, V},
  FourVector[mom1, li1] MetricTensor[li2, li3]];

AppendGenericCoupling[{V, V, V},
  FourVector[mom2, li2] MetricTensor[li3, li1]];

AppendGenericCoupling[{V, V, V},
  FourVector[mom3, li3] MetricTensor[li1, li2]];

AppendGenericCoupling[{S, S, V}, FourVector[mom3, li3]];

M$GenericCouplings = Join[M$GenericCouplings, {
    AnalyticalCoupling[s1 U[j1, mom1], s2 U[j2, mom2],
      s3 V[j3, mom3, {li3}], s4 V[j4, mom4, {li4}]] ==
     G[1][s1 U[j1], s2 U[j2], s3 V[j3],
       s4 V[j4]].{MetricTensor[li3, li4]},
```

```
    AnalyticalCoupling[s1 U[j1, mom1], s2 U[j2, mom2],
     s3 S[j3, mom3], s4 S[j4, mom4]] ==
     G[1][s1 U[j1], s2 U[j2], s3 S[j3], s4 S[j4]].{1}
    }];

Clear[GenericCouplingTypeQ, GenericCouplingIndex,
 AppendGenericCoupling];
```

The file SM_nlg.mod contains the modifications to the existing Standard Model interaction vertices and the new terms that appear:

```
GetCouplingIndex[coupling_] :=
 Position[M$CouplingMatrices,
  Cases[M$CouplingMatrices, coupling == _][[1]]][[1, 1]]

ReplaceCoupling[coupling_, pos_List, item_] :=
  M$CouplingMatrices[[GetCouplingIndex[coupling],
    2, pos[[1]], pos[[2]]]] = item

ReplaceCoupling[C[-V[3], V[3], V[2], V[2]], {2, 1},
  -I EL^2 CW^2/SW^2*(-1 + Gbeta^2/GaugeXi[W])];

ReplaceCoupling[C[-V[3], V[3], V[2], V[2]], {3, 1},
  -I EL^2 CW^2/SW^2*(-1 + Gbeta^2/GaugeXi[W])];

ReplaceCoupling[C[-V[3], V[3], V[1], V[2]], {2, 1},
  I EL^2 CW/SW*(-1 + Galpha Gbeta/GaugeXi[W])];

ReplaceCoupling[C[-V[3], V[3], V[1], V[2]], {3, 1},
  I EL^2 CW/SW*(-1 + Galpha Gbeta/GaugeXi[W])];

ReplaceCoupling[C[-V[3], V[3], V[1], V[1]], {2, 1},
  -I EL^2*(-1 + Galpha^2/GaugeXi[W])];

ReplaceCoupling[C[-V[3], V[3], V[1], V[1]], {3, 1},
  -I EL^2*(-1 + Galpha^2/GaugeXi[W])];

ReplaceCoupling[C[S[1], S[1], S[2], S[2]], {1, 1},
  -I EL^2/(4 SW^2 MW^2)*(MH^2 + 2 Gepsilon^2 MZ^2 GaugeXi[Z])];

ReplaceCoupling[C[S[1], S[1], S[2], S[2]], {1, 2},
  -I EL^2/(4 SW^2 MW^2)*(MH^2 (2 dZe1 - 2 dSW1/SW + dMHsq1/MH^2
    + EL/(2 SW MW MH^2) dTad1 - dMWsq1/MW^2 + dZH1 + dZG01))];
```

```
ReplaceCoupling[C[S[1], S[1], S[3], -S[3]], {1, 1},
  -I EL^2/(4 SW^2 MW^2)*(MH^2 + 2 Gdelta^2 MW^2 GaugeXi[W])];

ReplaceCoupling[C[S[1], S[1], S[3], -S[3]], {1, 2},
  -I EL^2/(4 SW^2 MW^2)*(MH^2 (2 dZe1 - 2 dSW1/SW + dMHsq1/MH^2
    + EL/(2 SW MW MH^2) dTad1 - dMWsq1/MW^2 + dZH1 + dZGp1))];

ReplaceCoupling[C[S[1], S[1], S[3], -S[3]], {1, 1},
  -I EL^2/(4 SW^2 MW^2)*(MH^2 + 2 Gkappa^2 MW^2 GaugeXi[W])];

ReplaceCoupling[C[S[1], S[1], S[3], -S[3]], {1, 2},
  -I EL^2/(4 SW^2 MW^2)* MH^2 (2 dZe1 - 2 dSW1/SW + dMHsq1/MH^2
    + EL/(2 SW MW MH^2) dTad1 - dMWsq1/MW^2 + dZG01 + dZGp1)];

ReplaceCoupling[C[S[1], S[2], S[2]], {1, 1},
  -I EL/(2 SW MW)*(MH^2 + 2 Gepsilon MZ^2 GaugeXi[Z])];

ReplaceCoupling[C[S[1], S[2], S[2]], {1, 2},
  -I EL/(2 SW MW)*(MMH^2 (dZe1 - dSW1/SW + dMHsq1/MH^2
    + EL/(2 SW MW MH^2) dTad1 - dMWsq1/(2 MW^2) + dZH1/2 + dZG01))];

ReplaceCoupling[C[S[3], S[1], -S[3]], {1, 1},
  -I EL/(2 SW MW)*(MH^2 + 2 Gepsilon MZ^2 GaugeXi[Z])];

ReplaceCoupling[C[S[3], S[1], -S[3]], {1, 2},
  -I EL/(2 SW MW)*(MMH^2 (dZe1 - dSW1/SW + dMHsq1/MH^2
    + EL/(2 SW MW MH^2) dTad1 - dMWsq1/(2 MW^2) + dZH1/2 + dZG01))];

AppendCoupling[c_, item_] := Module[{idx},
  idx = GetCouplingIndex[c];
  M$CouplingMatrices[[idx, 2]] =
   Append[M$CouplingMatrices[[idx, 2]], item]
  ]

AppendCoupling[C[V[1], -V[3], V[3]], -I EL* {0, 0}];

AppendCoupling[C[V[1], -V[3], V[3]],
 -I EL*{-Galpha/GaugeXi[W], 0}];

AppendCoupling[C[V[1], -V[3], V[3]],
 -I EL*{+Galpha/GaugeXi[W], 0}];
```

```
AppendCoupling[C[V[2], -V[3], V[3]],
 I EL CW/SW*{0, 0}];

AppendCoupling[C[V[2], -V[3], V[3]],
 I EL CW/SW*{-Gbeta/GaugeXi[W], 0}];

AppendCoupling[C[V[2], -V[3], V[3]],
 I EL CW/SW*{+Gbeta/GaugeXi[W], 0}];


ReplaceCoupling[C[S[1], -S[3], V[3], V[2]], {1, 1},
-I EL^2/(2 CW)*(1 + CW^2/SW^2 Gbeta Gdelta)];

ReplaceCoupling[C[S[1], S[3], -V[3], V[2]], {1, 1},
-I EL^2/(2 CW)*(1 + CW^2/SW^2 Gbeta Gdelta)];

ReplaceCoupling[C[S[1], S[3], -V[3], V[1]], {1, 1},
-I EL^2/(2 SW)*(1 - Galpha Gdelta)];

ReplaceCoupling[C[S[1], -S[3], V[3], V[1]], {1, 1},
-I EL^2/(2 SW)*(1 - Galpha Gdelta)];

ReplaceCoupling[ C[S[3], S[2], V[2], -V[3]], {1, 1},
EL^2/(2 CW)*(1 + CW^2/SW^2 Gbeta Gkappa)];

ReplaceCoupling[ C[-S[3], S[2], V[2], V[3]], {1, 1},
EL^2/(2 CW)*(1 + CW^2/SW^2 Gbeta Gkappa)];

ReplaceCoupling[ C[S[3], S[2], V[1], -V[3]], {1, 1},
EL^2/(2 SW)*(1 - Galpha Gkappa)];

ReplaceCoupling[ C[-S[3], S[2], V[1], V[3]], {1, 1},
EL^2/(2 SW)*(1 - Galpha Gkappa)];


AppendCoupling[C[S[2], S[1], V[1]], {0, 0}];

AppendCoupling[C[-S[3], S[3], V[1]], {0, 0}];

AppendCoupling[C[-S[3], S[3], V[2]], {0, 0}];

AppendCoupling[C[S[2], S[1], V[2]], EL/(2 CW SW)* {Gepsilon, 0}];
```

```
AppendCoupling[C[S[3], S[1], -V[3]], -I EL/(2 SW)* {Gdelta, 0}];

AppendCoupling[C[-S[3], S[1], V[3]], I EL/(2 SW)* {Gdelta, 0}];

AppendCoupling[C[S[3], S[2], -V[3]], EL/(2 SW)* {Gkappa, 0}];

AppendCoupling[C[-S[3], S[2], V[3]], EL/(2 SW)* {Gkappa, 0}];

ReplaceCoupling[C[-S[3], V[3], V[2]], {1, 1},
  -I EL MW SW/CW*(1 + CW^2/SW^2 Gbeta)];

ReplaceCoupling[C[S[3], -V[3], V[2]], {1, 1},
  -I EL MW SW/CW*(1 + CW^2/SW^2 Gbeta)];

ReplaceCoupling[C[-S[3], V[3], V[1]], {1, 1},
  -I EL MW*(1 - Galpha)];

ReplaceCoupling[C[S[3], -V[3], V[1]], {1, 1},
  -I EL MW*(1 - Galpha)];

ReplaceCoupling[C[-U[3], U[3], V[1]], {2, 1},
  I EL/Sqrt[GaugeXi[W]]*Galpha];

ReplaceCoupling[C[-U[4], U[4], V[1]], {2, 1},
  -I EL/Sqrt[GaugeXi[W]]*Galpha];

ReplaceCoupling[C[-U[3], U[3], V[2]], {2, 1},
  -I EL CW/SW/Sqrt[GaugeXi[W]]*Gbeta];

ReplaceCoupling[C[-U[4], U[4], V[2]], {2, 1},
  I EL CW/SW/Sqrt[GaugeXi[W]]*Gbeta];

ReplaceCoupling[C[-U[3], U[2], V[3]], {2, 1},
  -I EL CW/SW/Sqrt[GaugeXi[W]]*Gbeta];

ReplaceCoupling[C[-U[4], U[2], -V[3]], {2, 1},
  I EL CW/SW/Sqrt[GaugeXi[W]]*Gbeta];

ReplaceCoupling[C[-U[3], U[1], V[3]], {2, 1},
  I EL/Sqrt[GaugeXi[W]]*Galpha];

ReplaceCoupling[C[-U[4], U[1], -V[3]], {2, 1},
  -I EL/Sqrt[GaugeXi[W]]*Galpha];
```

```
ReplaceCoupling[C[S[1], -U[2], U[2]], {1, 1},
  -I EL MZ Sqrt[GaugeXi[Z]]/(2 SW CW)*(1 + Gepsilon)];

ReplaceCoupling[C[S[1], -U[3], U[3]], {1, 1},
  -I EL MW Sqrt[GaugeXi[W]]/(2 SW)*(1 + Gdelta)];

ReplaceCoupling[C[S[1], -U[4], U[4]], {1, 1},
  -I EL MW Sqrt[GaugeXi[W]]/(2 SW)*(1 + Gdelta)];

ReplaceCoupling[C[S[2], -U[4], U[4]], {1, 1},
  EL MW Sqrt[GaugeXi[W]]/(2 SW)*(1 - Gkappa)];

ReplaceCoupling[C[S[2], -U[3], U[3]], {1, 1},
  -EL MW Sqrt[GaugeXi[W]]/(2 SW)*(1 - Gkappa)];

ReplaceCoupling[C[-S[3], -U[4], U[2]], {1, 1},
  I EL MW Sqrt[GaugeXi[W]]/(2 CW SW)*(SW^2 - CW^2 - Gkappa)];

ReplaceCoupling[C[-S[3], -U[4], U[2]], {1, 2},
  I EL MW Sqrt[GaugeXi[W]]/(2 CW SW)
   * (dZe1 (SW^2 - CW^2) + dSW1/(CW^2 SW)
   + dUZZ1 (SW^2 - CW^2) + (2 SW CW)/dUAZ1)];

ReplaceCoupling[C[S[3], -U[3], U[2]], {1, 1},
  I EL MW Sqrt[GaugeXi[W]]/(2 CW SW)
   * (SW^2 - CW^2 - Gkappa)];

ReplaceCoupling[C[S[3], -U[3], U[2]], {1, 2},
  I EL MW Sqrt[GaugeXi[W]]/(2 CW SW)
   * (dZe1 (SW^2 - CW^2) + dSW1/(CW^2 SW)
   + dUZZ1 (SW^2 - CW^2) + (2 SW CW)/dUAZ1)];

M$CouplingMatrices =
  Join[M$CouplingMatrices, {

    C[-U[4], U[1], V[1], -V[3]] ==
     -I EL^2/Sqrt[GaugeXi[W]]*{{Galpha}},

    C[-U[3], U[1], V[1], V[3]] ==
     -I EL^2/Sqrt[GaugeXi[W]]*{{Galpha}},

    C[-U[4], U[1], V[2], -V[3]] ==
```

```
 I EL^2 (CW/SW)/Sqrt[GaugeXi[W]]*{{Gbeta}},

C[-U[3], U[1], V[2], V[3]] ==
 I EL^2 (CW/SW)/Sqrt[GaugeXi[W]]*{{Gbeta}},

C[-U[4], U[2], V[1], -V[3]] ==
 I EL^2 (CW/SW)/Sqrt[GaugeXi[W]]*{{Galpha}},

C[-U[3], U[2], V[1], V[3]] ==
 I EL^2 (CW/SW)/Sqrt[GaugeXi[W]]*{{Galpha}},

C[-U[4], U[2], V[2], -V[3]] ==
 -I EL^2 (CW^2/SW^2)/Sqrt[GaugeXi[W]]*{{Gbeta}},

C[-U[3], U[2], V[2], V[3]] ==
 -I EL^2 (CW^2/SW^2)/Sqrt[GaugeXi[W]]*{{Gbeta}},

C[-U[3], U[3], V[3], -V[3]] ==
 -I EL^2/Sqrt[GaugeXi[W]]
  * {{Galpha + CW^2/SW^2*Gbeta}},

C[-U[4], U[4], V[3], -V[3]] ==
 -I EL^2/Sqrt[GaugeXi[W]]
  * {{Galpha + CW^2/SW^2*Gbeta}},

C[-U[3], U[4], V[3], V[3]] ==
 2 I EL^2/Sqrt[GaugeXi[W]]
  * {{Galpha + CW^2/SW^2*Gbeta}},

C[-U[4], U[3], -V[3], -V[3]] ==
 2 I EL^2/Sqrt[GaugeXi[W]]
  * {{Galpha + CW^2/SW^2*Gbeta}},

C[-U[3], U[3], V[1], V[1]] ==
 2 I EL^2/Sqrt[GaugeXi[W]]*{{Galpha}},

C[-U[4], U[4], V[1], V[1]] ==
 2 I EL^2/Sqrt[GaugeXi[W]]*{{Galpha}},

C[-U[3], U[3], V[1], V[2]] ==
 -I EL^2 (CW/SW)/Sqrt[GaugeXi[W]]*{{Galpha + Gbeta}},

C[-U[4], U[4], V[1], V[2]] ==
```

```
 -I EL^2 (CW/SW)/Sqrt[GaugeXi[W]]*{{Galpha + Gbeta}},

C[-U[3], U[3], V[2], V[2]] ==
 2 I EL^2 (CW^2/SW^2)/Sqrt[GaugeXi[W]]*{{Gbeta}},

C[-U[4], U[4], V[2], V[2]] ==
 2 I EL^2 (CW^2/SW^2)/Sqrt[GaugeXi[W]]*{{Gbeta}},

C[-U[2], U[2], S[1], S[1]] ==
 -I EL^2 Sqrt[GaugeXi[Z]]/(2 SW^2 CW^2)*{{Gepsilon}},

C[-U[2], U[2], S[2], S[2]] ==
 I EL^2 Sqrt[GaugeXi[Z]]/(2 SW^2 CW^2)*{{Gepsilon}},

C[-U[2], U[4], S[3], S[1]] ==
 I EL^2 Sqrt[GaugeXi[Z]]/(4 SW^2 CW)*{{Gepsilon}},

C[-U[2], U[3], -S[3], S[1]] ==
 I EL^2 Sqrt[GaugeXi[Z]]/(4 SW^2 CW)*{{Gepsilon}},

C[-U[2], U[4], S[3], S[2]] ==
 EL^2 Sqrt[GaugeXi[Z]]/(4 SW^2 CW)*{{Gepsilon}},

C[-U[2], U[3], -S[3], S[2]] ==
 -EL^2 Sqrt[GaugeXi[Z]]/(4 SW^2 CW)*{{Gepsilon}},

C[-U[4], U[1], -S[3], S[1]] ==
 I EL^2 Sqrt[GaugeXi[W]]/(2 SW)*{{Gdelta}},

C[-U[3], U[1], S[3], S[1]] ==
 I EL^2 Sqrt[GaugeXi[W]]/(2 SW)*{{Gdelta}},

C[-U[4], U[1], -S[3], S[2]] ==
 EL^2 Sqrt[GaugeXi[W]]/(2 SW)*{{Gkappa}},

C[-U[3], U[1], S[3], S[2]] ==
 -EL^2 Sqrt[GaugeXi[W]]/(2 SW)*{{Gkappa}},

C[-U[4], U[2], -S[3], S[1]] ==
 -I EL^2 Sqrt[
    GaugeXi[W]]/(4 SW^2 CW)*{{Gkappa + Gdelta (CW^2 - SW^2)}},

C[-U[3], U[2], S[3], S[1]] ==
```

```
      -I EL^2 Sqrt[
        GaugeXi[W]]/(4 SW^2 CW)*{{Gkappa + Gdelta (CW^2 - SW^2)}},

    C[-U[4], U[2], -S[3], S[2]] ==
     -EL^2 Sqrt[
        GaugeXi[W]]/(4 SW^2 CW) * {{Gdelta + Gkappa (CW^2 - SW^2)}},

    C[-U[3], U[2], S[3], S[2]] ==
     EL^2 Sqrt[GaugeXi[W]]/(4 SW^2 CW)
      * {{Gdelta + Gkappa (CW^2 - SW^2)}},

    C[-U[3], U[3], S[1], S[1]] ==
     -I EL^2 Sqrt[GaugeXi[W]]/(2 SW^2)*{{Gdelta}},

    C[-U[4], U[4], S[1], S[1]] ==
     -I EL^2 Sqrt[GaugeXi[W]]/(2 SW^2)*{{Gdelta}},

    C[-U[3], U[3], S[2], S[2]] ==
     -I EL^2 Sqrt[GaugeXi[W]]/(2 SW^2)*{{Gkappa}},

    C[-U[4], U[4], S[2], S[2]] ==
     -I EL^2 Sqrt[GaugeXi[W]]/(2 SW^2)*{{Gkappa}},

    C[-U[3], U[3], S[2], S[1]] ==
     EL^2 Sqrt[GaugeXi[W]]/(4 SW^2)*{{Gkappa - Gdelta}},

    C[-U[4], U[4], S[2], S[1]] ==
     -EL^2 Sqrt[GaugeXi[W]]/(4 SW^2)*{{Gkappa - Gdelta}},

    C[-U[3], U[3], S[3], -S[3]] ==
     I EL^2 Sqrt[GaugeXi[W]]/(4 SW^2)*{{Gkappa + Gdelta}},

    C[-U[4], U[4], S[3], -S[3]] ==
     I EL^2 Sqrt[GaugeXi[W]]/(4 SW^2)*{{Gkappa + Gdelta}},

    C[-U[3], U[4], S[3], S[3]] ==
     -I EL^2 Sqrt[GaugeXi[W]]/(2 SW^2)*{{Gkappa - Gdelta}},

    C[-U[4], U[3], -S[3], -S[3]] ==
     -I EL^2 Sqrt[GaugeXi[W]]/(2 SW^2)*{{Gkappa - Gdelta}}}
];

Clear[GetCouplingIndex, ReplaceCoupling, AppendCoupling];
```

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