

Vilnius university  
Faculty of physics  
Theoretical physics department

Saulius Valatka

ONE-LOOP TESTS OF NON-LINEAR GAUGE  
FIXING

Semester Paper  
(Theoretical physics and astronomy studies program)

Vilnius 2010

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Renormalization</b>	<b>3</b>
2.1	Divergences in loop diagrams . . . . .	3
2.2	Resolving the divergences . . . . .	7
2.3	Renormalized perturbation theory . . . . .	9
2.4	Renormalization group . . . . .	11
<b>3</b>	<b>Renormalization in FeynArts</b>	<b>13</b>
<b>4</b>	<b>One-loop tests of non-linear gauge fixing</b>	<b>14</b>
<b>5</b>	<b>Conclusions</b>	<b>17</b>

# 1 Introduction

The currently accepted theory of elementary particles and fundamental interactions - the Standard Model (SM), introduced and shaped in the 70s, has been very successful at explaining all observed and measured effects up till now. All currently observed elementary particles — leptons and hadrons — and the fundamental interactions — electromagnetic, weak and strong — have been successfully incorporated into the theory and thoroughly tested in particle colliders. To date, all experimental data has agreed with the predictions of the Standard Model with high precision.

The process of calculating cross sections and decay rates becomes very simple in principle with the introduction of *Feynman diagrams*, which are basically pictorial representations of the process being considered. After drawing all diagrams for a given process, algebraic expressions are constructed from the diagrams using a set of rules, called the *Feynman rules*. The expressions are then calculated straightforwardly, summed up, squared and expressed as cross sections or decay rates. Though the procedure is relatively simple, usually, the amount of algebra to be done is huge, especially if we want to consider quantum corrections by including loop diagrams. Therefore, the calculations are usually done using computer software packages, one notable example being FeynArts.

Another notable problem in calculating scattering amplitudes is that some expressions in higher order diagrams may contain divergences. They introduce not only conceptual problems in applying quantum field theory, but also computational difficulties. And even though the conceptual issues have eventually been resolved and understood, the computational difficulties are still a major headache when doing calculations by hand. The process of getting rid of and interpreting the divergences is called *renormalization*. The renormalization procedure yet again encourages the use of various software packages, which nowadays are able to perform this most of the time technical task almost automatically.

One of the aims of this paper is to discuss the process of renormalization — its meaning in quantum field theories and its technical issues. We then discuss how the procedure is implemented in the FeynArts package and show some explicit one-loop calculations within the non-linearly gauge fixed Standard Model. The goal of the calculations is to show that FeynArts implements renormalization correctly, i.e. we don't get any dependence of the amplitudes on the gauge fixing parameters.

## 2 Renormalization

The need for a renormalization procedure arises when one considers higher order Feynman diagrams containing loops. In this section we will study the unbroken  $U(1)_{em}$  sector of the Standard Model, i.e. QED, and see how divergences arise in simple higher order diagrams. We first discuss how they can be understood and treated. Later we develop a more systematic approach to the renormalization procedure.

### 2.1 Divergences in loop diagrams

Consider a simple loop diagram in Fig. 1, the electron self-energy diagram.

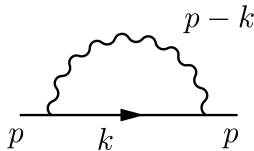


Figure 1: The electron self-energy diagram

The amplitude for it is given by

$$-i\Sigma^{[2]}(\not{p}) = (-ie)^2 \int \gamma^\nu \frac{-ig_{\mu\nu}}{k^2} \frac{i}{\not{p} - \not{k} - m} \gamma^\mu \frac{d^4k}{(2\pi)^4}, \quad (1)$$

where the superscript <sup>[2]</sup> indicates that this is a second order process. Looking at the integral it is immediately obvious that it is divergent, since there are three only powers of  $k$  in the denominator (even though it looks as a linear divergence, due to gauge invariance it is actually logarithmic [1]). At first sight this seems to be a disaster as we are calculating a correction to the tree level amplitude, but let's keep going. We define a *one-particle irreducible (1PI)* diagram as a diagram that cannot be split into two parts by removing a single line. The amplitude for the sum of all such diagrams is denoted by  $-i\Sigma$ , the absence of a superscript indicating that it is the 1PI electron self-energy with all orders included. To second order we have  $\Sigma^{[2]} = \Sigma$ . Any two-point diagram can then be expressed as a sum of such 1PI diagrams, as show in Fig. 2.

The total electron self-energy can now be written as a simple series

$$\frac{i(\not{p} + m_0)}{p^2 - m_0^2} + \frac{i(\not{p} + m_0)}{p^2 - m_0^2} (-i\Sigma(\not{p})) \frac{i(\not{p} + m_0)}{p^2 - m_0^2} + \dots \quad (2)$$

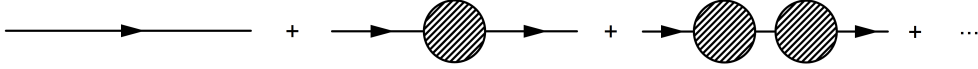


Figure 2: The full electron two-point function as a sum of 1PI diagrams

By noting that  $\Sigma$  commutes with  $\not{p}$  we can simplify the expression to

$$\frac{i}{\not{p} - m_0} + \frac{i}{\not{p} - m_0} \left( \frac{\Sigma(\not{p})}{\not{p} - m_0} \right) + \frac{i}{\not{p} - m_0} \left( \frac{\Sigma(\not{p})}{\not{p} - m_0} \right)^2 + \dots, \quad (3)$$

which we immediately recognize as a simple geometric series and can formally sum up. The resulting expression looks just like the electron propagator with a shifted electron mass, which makes sense, since it is unreasonable to think that quantum fluctuations will not shift it. Hence we introduce the electron propagator with full quantum corrections

$$S_F(p) = \frac{i}{\not{p} - m_0 - \Sigma(\not{p})}. \quad (4)$$

It is now obvious that the Lagrangian parameter  $m_0$  is *not* the physically measured electron mass, hence the subscript 0. The quantity  $m_0$  is referred to as the *bare mass*. The physical mass is defined as the pole of the propagator and is given by the solution of the equation

$$m_p - m_0 - \Sigma(m_p) = 0. \quad (5)$$

We can now try to calculate the mass shift to second order:

$$\delta m = m_p - m_0 = \Sigma^{[2]}(m_p) \approx \Sigma^{[2]}(m_0). \quad (6)$$

Calculating this shift requires performing the integral in (1), which is not an easy task. The first and most obvious problem is that it is divergent. The procedure of making it finite is called *regularization* and usually involves introducing some kind of momentum cut-off  $\Lambda$ . A notable example is the Pauli-Villars procedure which introduces massive particles to the theory. The cut-off can be justified by acknowledging the fact that our theory is not well defined for high momenta — we are parameterizing our ignorance. We hope that in the end a limit sending the cut-off to infinity can be taken, which gives sensible results. With the integral regularized it is only a matter of skill to do it. The usual procedure is to introduce Feynman parameters to combine the denominators, Wick rotate the integration plane and do the

simple resulting integral [2]. After performing the procedure and sending the cut-off to infinity the mass shift is

$$\lim_{\Lambda \rightarrow \infty} \delta m = \frac{3\alpha}{4\pi} \log \left( \frac{\Lambda^2}{m_0^2} \right). \quad (7)$$

A quantity which was supposed to be a small correction has turned out to be infinite.

Looking at the corrected electron propagator (4) we can see that not only is the pole shifted, but so is the residue. This shift can be interpreted as a rescaling of the field

$$\psi \rightarrow Z_2 \bar{\psi}. \quad (8)$$

Close to the pole the denominator has the form

$$(\not{p} - m) \left( 1 - \frac{d\Sigma}{d\not{p}} \Big|_{\not{p}=m} \right) + \mathcal{O}((\not{p} - m)^2) \quad (9)$$

and  $Z_2$  is given by

$$Z_2 \approx 1 + \frac{d\Sigma}{d\not{p}} \Big|_{\not{p}=m}, \quad (10)$$

thus the second order shift of the field normalization constant is

$$\delta Z_2 = Z_2 - 1 = \frac{d\Sigma^{[2]}}{d\not{p}} \Big|_{\not{p}=m}. \quad (11)$$

This quantity also turns out to be divergent when the cut-off is taken to infinity.

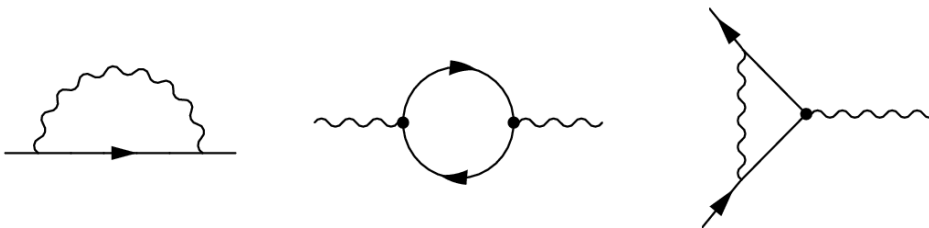


Figure 3: Divergent second order diagrams in QED

Further analysis of second order diagrams reveals that there are three different divergent diagrams in QED (c.f. [5]), all of which are shown in Fig. 3. The first diagram is the already familiar electron self-energy diagram. The

second diagram is referred to as the *vacuum polarization diagram*. It is given by

$$i\Pi_2^{\mu\nu} \equiv (-ie)^2(-1) \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \gamma^\mu \frac{i}{\not{k} - m} \gamma^\nu \frac{i}{\not{k} + \not{q} - m} \right]. \quad (12)$$

In analogy with the electron self-energy case, the sum of all 1PI diagrams of this kind is denoted as  $\Pi^{\mu\nu}(p)$ . The Ward identity restricts the tensor structure of this quantity so that we can factor out a projector[2]:

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \Pi(q^2). \quad (13)$$

The same reasoning as with the electron self-energy leads to a corrected expression for the photon propagator

$$D_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2(1 - \Pi(q^2))} + q_\mu q_\nu \text{ terms}, \quad (14)$$

where the second part is irrelevant for S-matrix elements, since it can be shown to cancel out using the Ward identity. Since  $\Pi(q^2)$  has no poles at  $q^2 = 0$ , that implies that the photon remains massless to all orders in perturbation theory. The residue of the pole at  $q^2 = 0$  is

$$Z_3 \equiv \frac{1}{1 - \Pi(0)}. \quad (15)$$

Since this factor always enters twice for every interaction involving an exchange of a photon, it is reasonable to define it as a charge shift

$$e \rightarrow Z_3^{1/2} e. \quad (16)$$

This shift also turns out to be infinite.

The last divergent diagram in Fig. 3 is known as the *vertex correction diagram*. In general when the vertex triangle is replaced with the sum of all 1PI diagrams, the resulting amplitude is denoted as  $\Gamma^\mu(p', p)$ , where  $p'$  is the momentum of the outgoing electron and  $p$  is the momentum of the incoming electron (the photon momentum is determined by the conservation of momentum). Then to second order we have

$$\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu, \quad (17)$$

where  $\delta\Gamma^\mu$  is the amplitude for the divergent second order diagram given in Fig. 3. Simple considerations of symmetry restrict the form of  $\Gamma^\mu$  to

$$\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2), \quad (18)$$

where  $q$  is the momentum of the outgoing photon and  $F_1(q^2)$  and  $F_2(q^2)$  are called *form factors*. As expected, infinities arise when one tries to calculate the shifts in these form factors. While  $\delta F_2(q^2)$  is actually finite, the divergence in  $\delta F_1(q^2)$  pops out in the worst place possible - while evaluating  $\delta F_1(0)$ , which should be zero.

## 2.2 Resolving the divergences

The last chapter revealed that many divergent quantities arise in QED when one goes beyond tree-level calculations. The corrections, which should be small in principle, turn out to be infinite, which casts doubts on the theory as a whole. However, it turns out that if one carefully defines the quantities being calculated and expresses them in physically measurable quantities only, the result will always be finite - in that sense QED is a renormalizable theory.

When calculating the shift of the electron mass in one-loop level, we discovered that it is shifted by an infinite quantity. At first sight, this seems absurd, however, we deal with such infinities in classical physics all the time when we consider point sources, e.g. a point electric charge. In classical physics we are well aware that the concept of a point charge is somewhat artificial and that's why we tend to ignore the resulting infinities. However, since quantum field theory is more fundamental in that sense, it is reasonable to ask whether an infinite bare mass is acceptable. An analogy can be drawn to solid state physics, where a concept of an effective mass is introduced as a means to include various interaction effects in solids. The physical field mass is in that sense also an effective mass, which is a result of various self-interactions of the fields. The main difference is that in solid state physics it is possible to measure the bare electron mass and then calculate the effective mass from first principles. In field theory, on the other hand, we cannot step out of the physical vacuum - the bare mass is not a measurable quantity in principle, hence the fact that it is infinite is only a technicality. Once one defines all quantities only in physically measurable parameters, all the infinities drop out.

The fact that it is not possible to step outside of the vacuum, which is a very active environment with particles popping in and out of existence, forces one to carefully define the quantities being calculated. In free field theory, a propagator is defined as a correlation function

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \frac{i}{\not{p} - m_0}, \quad (19)$$

where  $|0\rangle$  is the free field vacuum state. Obviously, the state of an interacting field vacuum is not the same as for a free field, hence the propagator should

be defined as

$$\langle \Omega | T\psi(x)\bar{\psi}(y) | \Omega \rangle. \quad (20)$$

As we already discussed, the interactions of fields produce a mass shift. Since we still expect the propagator to be an orthogonal wave, the only other possible modification is introducing a scaling factor

$$S_F \approx \frac{iZ}{\not{p} - m}. \quad (21)$$

It's not hard to guess that  $Z$  is the same scaling factor we got in (8). This result is generalized by the *LSZ reduction formula*, which relates Feynman diagrams and S-matrix elements. It states that a matrix element for a particular process is the amplitude of the *amputated* (i.e. external legs have no self-interactions) Feynman diagram for the process times a square root of a scaling factor for each external fermion field. The theorem basically factors out the field self-interactions to scaling factors.

It is now clear that we did not consider the external field corrections when discussing the electron-photon interaction vertex amplitude (17). Applying the LSZ reduction formula we have

$$Z_2\Gamma^\mu = (1 + \delta Z_2)(\gamma^\mu + \delta\Gamma^\mu) = \gamma^\mu + \delta\Gamma^\mu + \gamma^\mu\delta Z_2, \quad (22)$$

which in turn means that the first form factor shift has an extra term:

$$F_1(q^2) = 1 + \delta F_1(q^2) + \delta Z_2. \quad (23)$$

We stated that both shifts  $\delta F_1(q^2)$  and  $\delta Z_2$  are infinite in the previous chapter, but surprisingly enough, it turns out that their divergences both cancel each other, i.e.

$$\delta F_1(0) = -\delta Z_2, \quad (24)$$

hence

$$F_1(q^2) = 1 + [\delta F_1(q^2) - \delta F_1(0)] \quad (25)$$

is a finite and small correction to the first form factor.

The last divergence studied in the previous chapter is related to the rescaling of the elementary charge. We can dismiss the fact that the bare charge is infinite based on the same considerations of what is physically measurable and what is not. And since only the physical charge is measurable, we can always impose renormalization conditions for the charge, which will ensure that all physically measurable quantities turn out finite.

It is also worth mentioning that QED suffers not only from ultraviolet, but also *infrared divergences*[3]. These can also be treated by carefully defining

the physical process being examined, e.g. one has to consider the emission of external photons of very large wavelength, that can't be detected because of the experimental threshold. Divergences of this type arise because of the fact that the photon is massless.

## 2.3 Renormalized perturbation theory

In the previous chapter we gave a detailed explanation for each of the divergences that arise in QED and ways to treat them. The way of dealing with divergences we adopted is usually referred to as *bare perturbation theory*. While the outlined procedures give a better view of the cancellations involved, most of the time it is a tedious task that requires much attention to details. A more practical approach is *renormalized perturbation theory*, which deals with physical quantities from the start and imposes *renormalization conditions* that assure that the result will turn out finite and consistent.

The procedure consists of a few simple steps. First we rescale the fields so as to absorb the field renormalizations:

$$\psi = Z_2^{-1/2} \psi_0, \quad (26)$$

$$A^\mu = Z_3^{-1/2} A_0^\mu, \quad (27)$$

where the fields with subscript 0 indicate the bare fields. The Lagrangian now becomes

$$\mathcal{L} = Z_2 \bar{\psi} (i\not{\partial} - m_0) \psi - e_0 Z_2 Z_3^{1/2} \bar{\psi} \gamma^\mu \psi A_\mu - \frac{1}{4} Z_3 (F^{\mu\nu})^2. \quad (28)$$

We can introduce another scaling factor for the electric charge

$$e_0 Z_2 Z_3^{1/2} = e Z_1, \quad (29)$$

where  $e$  is the physical charge measured at large distances, i.e.  $q = 0$ . Comparing this definition with (16) we see that they are consistent with each other only if  $Z_2 = Z_1$ . This actually turns out to be true as a case of the Ward identity and it suggests that the electron charge shift depends only on the renormalization of the photon field and is not a property of the electron. If that were not the case, introducing other charged particles to the theory would reveal that each particle has its own elementary charge, since they would be shifted differently.

By introducing another set of variables

$$\delta_1 = Z_1 - 1, \delta_2 = Z_2 - 1, \delta_3 = Z_3 - 1 \quad (30)$$

and

$$\delta_m = Z_2 m_0 - m, \quad (31)$$

we can rewrite the Lagrangian as

$$\begin{aligned} \mathcal{L} = & \bar{\psi}(i\not{\partial} - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}(F^{\mu\nu})^2 \\ & + \bar{\psi}(i\delta_2\not{\partial} - \delta_m)\psi - e\delta_1\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}\delta_3(F^{\mu\nu})^2. \end{aligned} \quad (32)$$

We got rid of the bare parameters completely at the cost of introducing three new terms, called the counter terms, which absorb the infinite and unobservable shifts. These terms appear as three new types of interactions in Feynman diagrams as shown in Fig. 4.

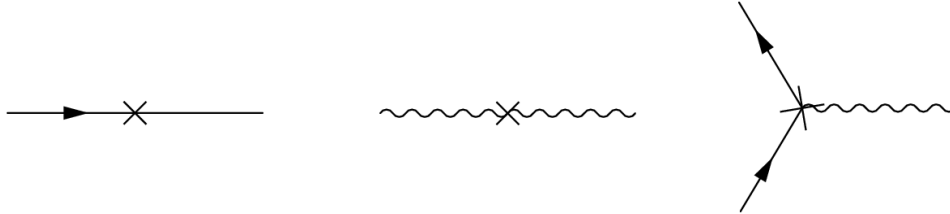


Figure 4: Counter term diagrams for QED

Their amplitudes can be read off the Lagrangian, i.e. the electron propagator counter term is given by

$$i(\not{p}\delta_2 - \delta_m), \quad (33)$$

the photon counter term is

$$-i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3 \quad (34)$$

and the electron-photon vertex counter term is

$$-ie\gamma^\mu\delta_1. \quad (35)$$

When drawing diagrams for a process one has to include the counter terms as well.

Last but not least we have to specify the renormalization conditions, which define the physical masses and electric charge and keep the field normalizations equal to 1. These are basically the self-consistency conditions discussed in the previous chapter, i.e. ensuring that the pole of the propagator is the physical mass etc. Since we are now discussing renormalized

perturbation theory, the quantities  $\Sigma(\not{p})$ ,  $\Pi(q^2)$  and  $\Gamma(p', p)$  are now defined to include the counter term diagrams as well as physical masses and charge. The renormalization conditions can then be stated as

$$\Sigma(\not{p} = m) = 0, \quad (36)$$

$$\left. \frac{d}{d\not{p}} \Sigma(\not{p}) \right|_{\not{p}=m} = 0, \quad (37)$$

$$\Pi(q^2 = 0) = 0, \quad (38)$$

$$-ie\Gamma^\mu(p' = p = 0) = -ie\gamma^\mu. \quad (39)$$

These relations can then be used to calculate the parameters  $\delta_1$ ,  $\delta_2$ ,  $\delta_3$  and  $\delta_m$  in terms of the physical parameters at a specific order by doing the loop integrals explicitly. In the end the result will be finite and expressed in the physical parameters only.

## 2.4 Renormalization group

When discussing the photon self-energy diagram in the previous chapter we derived the corrected photon propagator (14), which we repeat here for convenience:

$$D_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2(1 - \Pi(q^2))} + q_\mu q_\nu \text{ terms}. \quad (40)$$

This correction can be interpreted as a shift of the electron charge, which is the coupling constant, since a factor of  $e$  lies at each end of the photon propagator. Thus we conclude that the parameter which was supposed to be a constant is actually a  $q$  dependent quantity

$$\alpha_0 \rightarrow \alpha_{eff}(q^2) = \frac{e_0^2/4\pi}{1 - \Pi(q^2)}. \quad (41)$$

Physically the  $q$  dependence of the electric charge can be explained by the effect of screening. In classical electrodynamics we are well aware of the fact that a medium can alter the strength of the electric field, which in turn can be interpreted as an alteration of the charge. This alteration is due to the screening effects of the charged particles in the medium. We expect a similar effect in the vacuum where electron-positron pairs can be created spontaneously and screen the electromagnetic interaction - an effect known as *vacuum polarization*. Far away from a test charge the vacuum is a dielectric medium, hence the strength of the interaction seems smaller. The interaction should increase as we near the test charge and enter the electron-positron cloud, since then we are able to see the real charge. Indeed, in

one-loop level the integral for  $\Pi_2(q^2)$  can be done in the limit  $-q^2 \gg m^2$  yielding

$$\hat{\Pi}_2(q^2) = \frac{\alpha}{3\pi} \left[ \log \left( \frac{-q^2}{m^2} \right) - \frac{5}{3} + \mathcal{O} \left( \frac{m^2}{q^2} \right) \right], \quad (42)$$

where the hat indicates that the divergence is already subtracted. The effective coupling constant in this limit is

$$\alpha_{eff}(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \log \left( \frac{-q^2}{Am^2} \right)}, \quad (43)$$

where  $A = \exp(5/3)$ . As expected, the coupling gets stronger at small distances.

The fact that the coupling constant is not actually a constant is referred to as the *running of the coupling*. The proper tool for understanding this effect in particular and renormalization in general is the *renormalization group* approach. Since a detailed study of it is beyond the scope of this paper, we outline only the basic ideas here. The running of the couplings turns out to be closely related to the concept of scaling, which can be performed by scaling distance, i.e.

$$x \rightarrow xb, \quad (44)$$

or momenta

$$k \rightarrow k/b, \quad (45)$$

where  $b$  is the scaling factor. After performing a scaling operation, the initial Lagrangian of the theory can be brought back to its initial form by assuming that the parameters get shifted. In momentum space this corresponds to integrating out large momenta. By taking the limit  $b = 1$  one can extract a continuous dependence of the parameters on the scale. This is exactly what we got in case of the QED coupling constant. Evolution of the coupling constants is described by the beta functions of the theory, defined as

$$\beta = \frac{M}{\delta M} \delta \lambda, \quad (46)$$

where  $M$  is the energy scale and  $\lambda$  is a coupling constant.

A detailed discussion about the renormalization group approach can be found in the references given at the end of the paper.

### 3 Renormalization in FeynArts

Calculating quantities using Feynman diagrams by hand is a tedious task, hence it is almost always done using some computer software package. One of the most popular packages is *FeynArts*<sup>1</sup>, which runs on top of *Mathematica*, hence is available for a variety of platforms and operating systems [6].

FeynArts is actually only one part of the whole package, which also includes FormCalc and LoopTools. The basic structure of the calculation process is as follows: we first define the process we want to calculate by specifying the ingoing and outgoing particles. FeynArts is then responsible for finding all possible ways of connecting the incoming and outgoing particles, i.e. creating topologies of the soon to be Feynman diagrams. This includes counter terms and loop diagrams up to a specified level. Next, Feynman diagrams are generated by inserting appropriate particles into the topologies, which are allowed by the model being used. After the diagrams are completed, algebraic expressions are associated with the diagrams — this step is performed by FormCalc. Up till now, everything is being done in Mathematica. Algebraic simplification of the expressions usually requires a lot of computation power hence this step is being done internally in FormCalc by calling another program, called FORM, under the covers. Divergences are also taken care of during this step by calling various LoopTools subroutines where loop integrals are reduced to Passarino-Veltman integrals and the counter terms are evaluated using the defined renormalization scheme. In the end, FormCalc produces a FORTRAN program, which produces numerical results for the cross section given various process parameters, e.g. the center of mass energy of the process.

Renormalization in FeynArts is done almost automatically. All that needs to be specified are the counter term vertex expressions and the renormalization conditions. All of that can be specified in so called *model files*, which define the propagators, vertices and counter terms for a theory. Vertex expression specifications are split into two pieces. The first piece is defined in the so called *generic model file*, which defines the geometry of spacetime and specifies the particle types, i.e. scalars, spinors etc., and the ways they interact. The model file then specifies all the particles in the theory, e.g. electrons, photons etc. and defines the interaction vertices by specifying coefficients for all terms with different structures, e.g. a metric term, momentum product term etc.

---

<sup>1</sup><http://www.feynarts.de/>

For example, the QED model uses a generic model file `QED.mod`, which defines the structure of a vector-vector interaction as

```
AnalyticalCoupling[ s1 V[i, mom1, {li1}],
                    s2 V[j, mom2, {li2}] ] ==
G[1][s1 V[i], s2 V[j]] .
{ MetricTensor[li1, li2] ScalarProduct[mom1, mom2],
  MetricTensor[li1, li2],
  FourVector[mom1, li2] FourVector[mom2, li1] }
```

This means that a term for a vector-vector interaction in the most general case will be a sum of three factors proportional to  $(p_1 \cdot p_2)g^{\mu\nu}$ ,  $g^{\mu\nu}$  and  $p_1^\mu p_2^\nu$ . The model file then specifies these coefficients for specific vector particles at tree-level and one-loop level. So in QED we have

```
C[ V[1], V[1] ] == I * { {0, dZAA1}, {0, 0}, {0, -dZAA1} },
```

where the zeros in the first column indicate that there is no interaction of a photon with a photon in QED — that is simply the propagator and not an interaction. However we do have an interaction term in one-loop level. The constant `dZAA1` corresponds to the photon field renormalization constant  $\delta_3$ , so when we combine the structure constants from the generic model file with the coefficients, the expression reduces to

$$-i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3, \quad (47)$$

which is exactly what we got in (34). All that's left is specifying the renormalization conditions, which are expressed in simple expressions like

```
RenConst[ dZAA1 ] := FieldRC[V[1]],
```

which identifies the constant `dZAA1` as the photon field renormalization constant. All of the rest is taken care by the package. The same ideas apply to other models, only the expressions can become much more complicated.

## 4 One-loop tests of non-linear gauge fixing

Non-linear gauge fixing is implemented in FeynArts as an addition to the `Lorentz.gen` generic model file and the `SM.m` model file for the Standard Model [7, 8]. The main purpose of this implementation is checking the correctness of the software package itself, since no physical result should depend on any of the gauge fixing parameters and in the case of non-linear gauge fixing we have 5 extra parameters.

The first check is to determine whether the implementation is correct at all. The most obvious check is to see whether the new types of interactions appear in the list of diagrams produced for a particular process. We chose the process of an electron and positron becoming top anti-top quarks, or  $e\bar{e}t\bar{t}$  in short as the main process for our checks. Looking at the diagrams we immediately recognize the new type of interactions introduced by non-linear gauge fixing, e.g. vector-vector-ghost-ghost interactions. A few examples of such interactions are shown in Fig. 5.

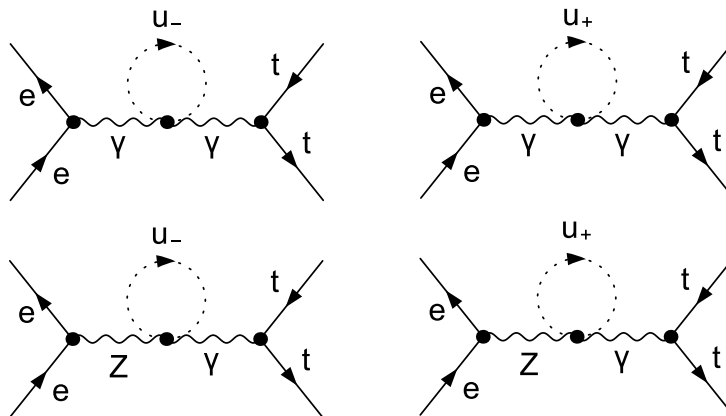


Figure 5: Vector-vector-ghost-ghost interaction examples

The next step is calculating an amplitude for some specific set of initial conditions so that the result would be a number. We chose a simple scattering set up with the process energy set to  $s = 500 \text{ GeV}$  and all particles being unpolarized. At first we tested the correctness of the results comparing them to the results of the same processes within the linearly gauge fixed Standard Model. All the results matched with unlimited accuracy. This can be easily understood having in mind that FormCalc does a lot of algebraic simplifications. At tree level it is possible to cancel the gauge fixing parameters completely, hence the results match exactly. That's why it is important to compare the results at one-loop level. In this case, when divergent quantities have to be regularized and treated, FormCalc is not able to carry out the cancelations algebraically and must rely on numerical methods. In this case some numerical fluctuations are to be expected. And that's exactly what we see when looking the dependences of a specific amplitude on the  $\alpha$  and  $\beta$  parameters (defined in [7, 8]) in Fig. 6. We see that when the parameters reach a magnitude of  $10^6$ , the change in the result is visible, yet numerically it is only  $\approx 0.002\%$ . Thus we can state that the implementation of non-linear gauge fixing is correct at least within one-loop level.

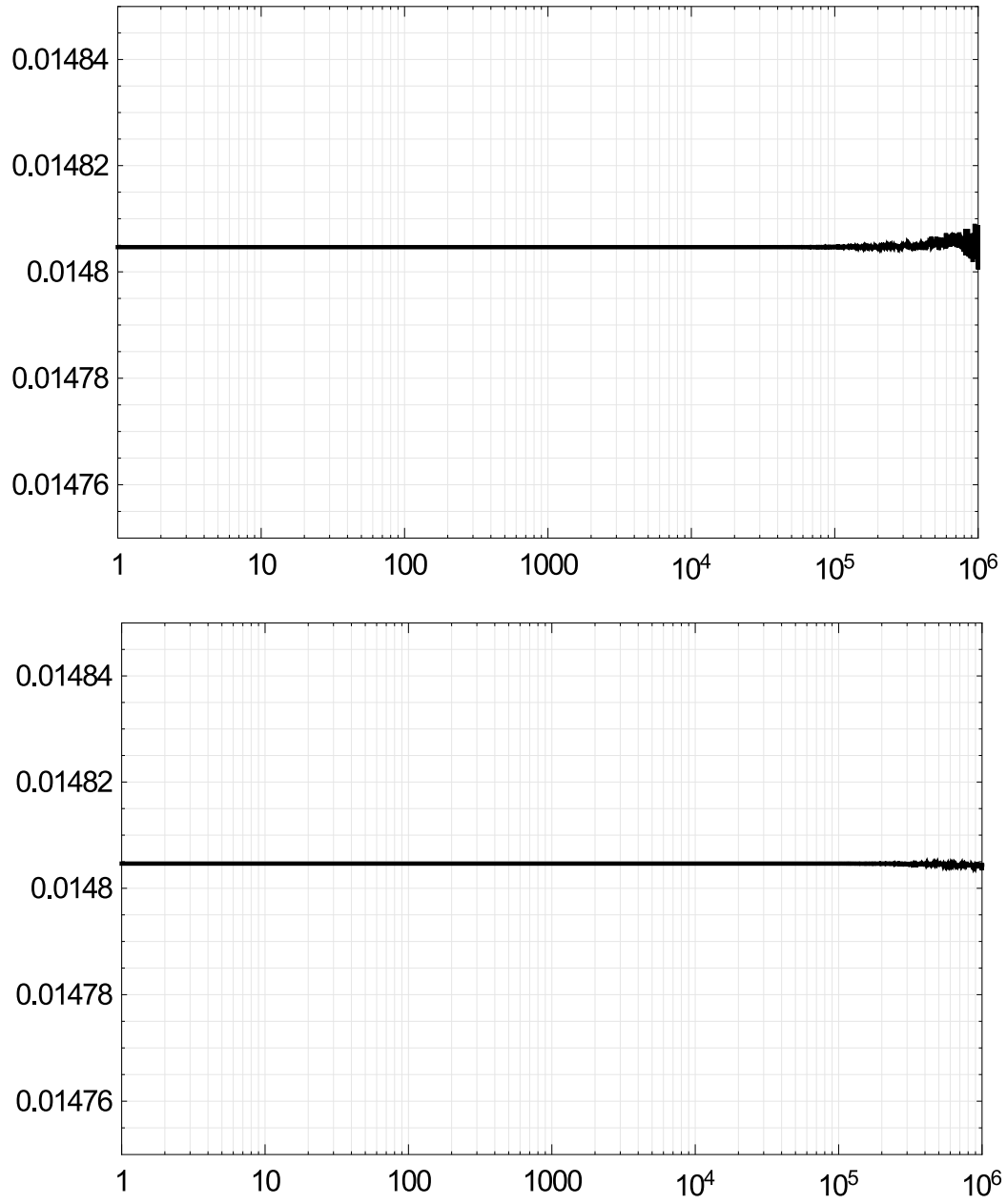


Figure 6: The dependence of the  $e\bar{e}t\bar{t}$  process amplitude at a specific energy on the  $\alpha$  and  $\beta$  non-linear gauge fixing parameters (defined in [7, 8])

## 5 Conclusions

In this paper we gave a brief review of the renormalization procedure in quantum field theories, in particular — QED. We discussed the problems that arise when one does not take care in defining the quantities being calculated and tries to do loop integrals at one-loop level explicitly. In the first part of the paper we showed that all of the arising divergences can be treated and we explained their origins. We then discussed a more systematic approach to renormalization by introducing renormalized perturbation theory and the renormalization group approach.

The second part of the paper dealt with renormalization in the FeynArts software package for calculating scattering amplitudes. We explained how FeynArts implements renormalization in general and then discussed the various tests of the non-linear gauge fixed Standard Model implementation within this package at one-loop level. Finally we reached a conclusion that the implementation is correct as the calculation errors were on the level of numerical fluctuations only.

## References

- [1] H. Aitchison, *Gauge theories in particle physics vol. 1*, (IOP, 2002)
- [2] M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, (Addison-Wesley, Reading, MA, 1995)
- [3] S. Weinberg, *The Quantum Theory of Fields Vol.2: Modern Applications*, (Cambridge Univ. Press, Cambridge, 1996)
- [4] F. Jegerlehner, *Renormalizing the Standard model*, PSI-PR-91-08, 1991.
- [5] A. Zee, *Quantum Field Theory in a Nutshell*, (Princeton University Press, 2003)
- [6] T. Hahn and M. Perez-Victoria, *Comput. Phys. Commun.* **118** (1999) 153; [hep-ph/9807565](#).
- [7] J. Pašukonis, *Implementation of non-linear gauge-fixing in FeynArts package*, [arXiv/0710.0159](#).
- [8] S. Valatka, *Non-linear gauge fixing in the Standard Model*, Semester paper 2009.